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Financial Pyramids in Transitional Economies A Game-Theoretic Approach

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We analyse the phenomenon of financial pyramids (Ponzi games) that were surprisingly widespread in transitional economies in the mid-1990s. Financial pyramids are modelled as a stochastic game under incomplete information between a Ponzi firm — an intended builder of the pyramid — and a population of heterogeneous individual investors. It is shown that although equilibrium strategies involving individual decision to invest may exist in stage-games, the whole dynamic game has no equilibrium except a trivial one, in which nobody invests, and thus the pyramid does not grow. Thus, individual decisions to invest are attributable to either naive belief in the firm's honesty or to an inappropriate specification of the dynamic optimization problem. Several plausible dynamics of population behaviour, especially those resulting from evolutionary games, are proposed and compared to the actual experience of Russia.

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NON-TECHNICAL SUMMARY

In this paper, we analyse the phenomenon of financial pyramids, or Ponzi games, paying special attention to its application to transitional economies. A financial pyramid is a firm which offers extremely attractive interest on private deposits, sustaining its credibility by paying the promised interest to initial debtholders with money brought in by late-comers. Because the interest payment is many times higher than the market rate, the pyramid is ultimately doomed to collapse, leaving ruined its current investors. This sort of "business" has been, of course, prohibited in developed countries since the early 1920s; somewhat surprisingly it emerged in the mid-1990s in a number of emerging market economies, especially in Central Europe and the former USSR. Because of the institutional vacuum caused by the changing economic and political system, no efficient barriers were put up against such pyramids. As a result, in Russia alone about US\$ 15 billion was stolen out of the pockets of some 20 million people.

But even besides institutional issues, this phenomenon is puzzling from a purely economic viewpoint: how could millions of individuals (conceived in economic theory as rational agents) come to an apparently irrational decision to invest in such a risky firm? This collective madness is indeed hard to explain if one stands within the conventional paradigm of substantive rationality. However, public interest in these pyramids can be rationalized using a weaker version of bounded rationality (Simon, 1955; 1978). In our model we consider two types of debtholders. Sophisticated individuals are those who know that the pyramid will eventually collapse, but they believe they can foresee when this will happen and withdraw their savings beforehand. Naïve individuals simply do not realize that the pyramid is cheating and invest in it simply because other people do and these people are observed receiving positive rewards. The behaviour of investors of this latter type can be conveniently described by the means and methods of evolutionary games.

We model financial pyramids as a stochastic game under incomplete information between a Ponzi firm — the intended builder of a pyramid — and a population of heterogeneous individual investors. In a simple example of a dynamic game with just two strategies in every stage-game, we first formulate a simple optimal stopping condition for the firm: the firm should keep its promises whenever the pyramid grows and defect once and forever afterwards. Then we analyse a truncated game of the firm against a single investor and show that this dynamic game has two

equilibrium components, one of which invokes investment up to some stage; moreover, these components are shown to survive most equilibrium refinements. This sort of game, however, is a mis-specification of the actual interaction between the population of individuals and the firm under incomplete information. To show that the strategy of "investing up to some stage, and withdrawing afterwards" (we call it *cautious*) is erroneous, we first build the full types and strategies' space of the stochastic game under incomplete information. We then prove by contradiction that the individuals' strategy in the only equilibrium of the full-scale Ponzi game of incomplete information consists of never investing in the pyramid.

The pyramid phenomena is thus determined as a disequilibrium phenomenon motivated by less-than-perfectly rational considerations attributable to either naive beliefs in a firm's honesty or to an improper specification of the dynamic optimization problem. The explicit game dynamics of the population's behaviour is considered and derived from both of the two above perspectives. Specifically, we consider evolutionary dynamics via the imitation of successful behaviour (Weibull, 1995), constructed without and with advertising, and a new variant of replicator dynamics derived from the logic of the Ponzi game. Equipped with these dynamics, we construct and plot specific discrete-time paths of a number of investors and the amounts of money attracted by the pyramids that are compatible with the actual experience of Russia.

1. BACKGROUND AND MOTIVATION

The story motivating this work is both sad and fascinating. Imagine a newly emerged bank or financial company — call it NNN, — which without a proper banking license has launched an extraordinary advertising campaign promising extremely high (4 – 6 times the market rate) interest on private deposits. Many individual investors recklessly deposit their money in NNN, allowing the firm to raise its liabilities to several million US\$ within a few months. Unfortunately, it turns out shortly thereafter that the company is unable to pay the promised interest, or to return its debt. This is of little surprise. By failing to raise money by means of profitable crediting activities, NNN, just as other similar firms, sustains its credibility by repaying the interest to its initial debtholders with money brought in by late-comers. The end of the story is as uniform as inevitable: NNN eventually becomes bankrupt, its managers "disappear," and its former creditors (nicknamed "diluted debtholders") find that the firm's remaining assets are worth nearly nothing. In other words, its managers have escaped with debtholders' money.

The above story of NNN is not a fairy tale nor a historical accident due to the controversial ingenuity of Carlo Ponzi in the 1920s. It became a reality of the 1990s, especially since 1994, when a large number of real financial pyramids flourished in a number of economies in transition, such as Russia, Ukraine, Bulgaria, Romania, Armenia, Albania — to name a few. The eventual collapse of the Ponzi schemes left several *million* people diluted of their savings and had far-reaching social, economic and political consequences, such as increasing public mistrust in the banking sector and in public authorities, impoverishing the population, creating social upsets and unrest, and initiating one revolution (in Albania)¹. To some extent, the current economic instability, as well as general vulnerability of young democracies in these countries, can be attributed to the distressing experience of financial pyramids.

These facts warrant the need to study financial pyramids as a phenomenon peculiar to economies in transition, for modern pyramids owe much of their specific features to the transformational patterns of emerging

¹ Our appeal to the practice of financial pyramid is motivated primarily by the case of Russia since it is the one we are most familiar with. However, the discussion that follows remains applicable to most other transitional economies, where pyramids' expansion seems to be driven by essentially the same forces.

market economies. This approach to financial pyramids should be contrasted with the concept of "Ponzi games" of modern macroeconomics. In this literature, the possibility of Ponzi games is usually ruled out on the grounds that one cannot borrow forever — the net discounted flow of borrowings should be nonpositive at the limit.² Closely related to this vein of research is an extensive literature on asset/money bubbles (Blanchard and Watson, 1982; Tirole, 1982, 1985; Weil, 1987; Rosser, 1991; Bhattacharya and Lipman, 1995; Werner, 1997; Gilles and LeRoy, 1997; Fukuta, 1998). The traditional rational expectations approach to bubbles justifies their possibility as a solution to the self-confirmed expectational difference equation, and these solutions usually emerge under particular conditions, such as myopic expectations or dynamic inefficiency. In recent years, this literature has expanded, incorporating further relaxations of the rational expectations assumption (Bertocchi and Wang, 1995; Goldberg and Frydman, 1996), and spreading over a broader class of socio-cultural phenomena (Cozzi, 1998; Orlan, 1994; Bikhchandani *et al.*, 1992), as well as empirical (Wu, 1997; Froot and Obstfeld, 1991) and experimental (Porter and Smith, 1995) analysis of particular bubbles, including the experience of the recent Asian crisis (*e.g.*, Whittaker and Kurosawa, 1998) and Ponzi schemes in transitional economies (Bhattacharya, 1999).

Our approach is somewhat different from those mentioned above. We treat Ponzi games as *games* in the game-theoretic sense, but also as "plays" or "gambles" of individual investors against a Ponzi firm, with our primary aim to describe the experience of Ponzi games in transitional economies. By stressing the economic character of our approach, we explicitly leave aside ethical or legal aspects of Ponzi games, as well as their socioeconomic and political origins and consequences. At the same time, the dynamics of the social interactions of individual investors, their "herd behaviour" (Banerjee, 1992), cannot be completely neglected even in an economic analysis of this phenomenon.³ Indeed, their behaviour appears to be a sort of "collective madness" with respect to issues pertinent to their own wealth, whereas their behaviour should have been rational! We consider two possible rationalizations of this seemingly irrational behaviour, which arise as *substantively rational* behaviour along the conventional Bayesian lines and *boundedly rational* behaviour within the frame of evolutionary games.

² King and Ferguson, 1993; O'Connell and Zeldes, 1988; Forslid, 1998 are a few examples of the relaxation of this principle, based on different premises.

³ Moreover, even some of these forms of apparent "madness" may have rationalistic roots (Bikhchandani *et al.*, 1992).

More fundamentally, we trust that the ultimate cause of the massive expansion of the Ponzi schemes lies in the peculiarities of social and institutional transformation. This massive expansion resulted in significant mental shocks for the citizens of the countries in transition. Imagine someone with 20 – 30 – 50 years of planned economy experience, living now under drastically changed rules for everyday economic activities,⁴ complemented by a jump in prices and massive impoverishment. Not surprisingly, many people felt quite uncomfortable in the new, unsecured circumstances, struggling simultaneously to survive to the extent of their abilities and understanding. Participation in the financial market games seemed one such survival strategy: at first glance it looked like an easy way to maintain one's living standard, where the Ponzi firm provides a substitute for the government as an "insurer" of individual welfare. Since the first investors seemingly succeeded, more and more people were about to find out that the idea was not bad at all — up to the moment where the pyramid's collapse made clear the fallacy of such hopes.

In this paper we formalize the above considerations in terms of a stochastic game and make some analytical conclusions. More detailed analysis of the origins of financial pyramids and markets in transitional economies is left for future research. The main analytical framework of discrete-time stochastic Ponzi games is developed in the next section. In Section 3 we build a parametric model of the Ponzi game and define the optimum strategy of the firm. In Section 4 we introduce two different sorts of individuals that we label as *naïve* and *sophisticated*. Our experience suggests that such separation is meaningful from a practical viewpoint, which warrants the use of different approaches when modelling their behaviour. In Section 4 we argue that sophisticated strategies are fallacious; this section contains the main equilibrium results about the Ponzi games. In the Appendix, we built the general types space for the stochastic game of incomplete information, concluding with a simple argument that shows that the only equilibrium of this game is the trivial one, in which the pyramid does not grow at all. This conclusion permits us to characterize the development of Ponzi schemes as a disequilibrium phenomenon, which nevertheless has its rationalistic origins. The dynamics of the pyramid's growth for both naïve and sophisticated individuals is developed in Section 5; the next section contains the results of the numerical simulations and several possible extensions of the basic dynamics. Conclusions and implications of the model are discussed in the last section.

⁴ In the early 1990s, some elderly people were really afraid of and furious at seeing identical loafs of bread sold in different shops at different prices!

2. PONZI GAME: GENERAL SETUP

From a microeconomic perspective, it is natural to view the Ponzi game as a stochastic game, an approach which was introduced by Shapley (1953) for the zero-sum case and by Sobel (1971) for general noncooperative games. These games are "truly dynamic" in the sense that the environment in which rational players make their decisions evolves as a function of the players' past decisions. This is exactly what happens in the case of financial pyramids. One player (the Ponzi firm) decides whether to stay in business or to escape with debtholders' money conditional upon the amount the debtholders brought to its premises (and upon what they are expected to bring in the future). This decision of the firm determines the investors' payoffs. Somewhat more formally, a stochastic Ponzi game Γ is a discrete-time sequence γ_t , $t = 0, 1, 2, \dots$, where each $\gamma_t \in \{\Gamma_1, \dots, \Gamma_s, \dots, \Gamma_S\}$ is one of a finite set of possible games played at time period t , where the game to be played in period t depends, in general, on the entire history of the players' decisions. Such decisions are made anew by all individuals and the firm in every period t , so that every Γ_s is a nonzero-sum,⁵ non-cooperative, simultaneous move game of all potential individual investors and the Ponzi firm. The set S is sometimes called state space (Sobel, 1971); since the state s at period t corresponds uniquely to the game Γ_s to be played at that time period, each $\gamma_t = \Gamma_s$ is also called a stage-game.

The stochastic Ponzi game Γ is defined by a septuple $\{N+1, S, Q, \gamma_0, u, \rho, \mu\}$, where the first six elements characterize a complete information game where all players know the payoffs of each other and observe the histories of past plays at every t . More specifically,

1. $N + 1$ are players indexed by F_t of which the first N are prospective individual investors, and the last one is the Ponzi firm. Throughout this paper we consider the case where a large *population* of N private individuals ($N < \infty$) who, acting independently from each other, play against a *single* financial company. This latter is somewhat loosely called a "firm," and which is the intended pyramid builder, undertaking no other activities apart from raising money in order to steal it.
2. S is the *state space* which uniquely determines a simultaneous-move stage game Γ_s for every time period. Assumption $|S| < \infty$ is natural in

⁵ The variant we construct below is a nonzero-sum game whose payoff structure emphasizes changes in players' wealth rather than their total wealth position. It is possible (though less convenient for our purposes) to redefine individuals' payoffs to obtain a zero-sum variant.

the present context since the set of players is finite, as is (see below) the set of strategies and payoffs (possible amounts of money attracted by the firm).

3. $Q = \prod_{\ell}^{N+1} Q_{\ell}$ is the set of all combinations of players' strategies. The set of *strategies* of player F , denoted by Q_F , is fixed for every player and every Γ_s ; without loss of substance we limit discussion to the case $|Q_F| = 2$. Of the first N players (individuals), every one at any moment of time can choose either an I-strategy, *i.e.*, to *Invest* in the firm, or a W-strategy, *i.e.*, to *Withhold* from such investment — formally, $Q_F = \{I, W\}$ for $F = 1, \dots, N$. The last player — the firm — can either Cooperate with investors by paying the promised interest and the principal amount upon request, or Defect,⁶ which is tantamount to the pyramid's collapse — thus, $Q_F = \{C, D\}$ for $F = N + 1$. Since the set N of individuals is finite, as is the strategies' space, the total number of strategies' profiles is finite⁷ as well. A typical profile of the set Q is denoted by q ; subscript t is applied when we need to emphasize the time period. In what follows, we use lowercase Latin letters q and p to indicate strategies' profiles; subscripts F , N or $N + 1$ always refer to players, and all other subscripts for q and p refer to probabilistic weights attached to pure strategies in mixed strategies' profiles. These uses should be clear from the context; for example, q_i stands for the probability of playing pure strategy i or, isomorphically, for the proportion of individuals in the population playing strategy i . Pure strategy i of player F is written as e_{iF} pure strategy of the population as e_i .

4. γ_0 is the *initial stage game*. We have to suppose that the firm has some initial capital endowment and that at least some individuals play I at this initial stage, leaving aside possible reasons for such an initial decision.

5. $u = \{u_F\}$ is a vector of individual *payoffs* for all F . In view of the sequential nature of the problem, we suppose that the payoff of every player for the whole game Γ is given by the stage-additive utility func-

⁶ We do not need to make a formal distinction between the "forced collapse" of the Ponzi scheme which has not been expected by the firm, and a "managed" or "voluntary" one, when the firm knew it will happen and has prepared to get out of the situation in advance. If the Ponzi firm behaves optimally (and we are interested in the case when it does), the only difference between the two cases will be its last period's payoff; there is no difference in the dynamics of the pyramid's growth.

⁷ More specifically, it is of cardinality 2^{N+1} , which is still a very large number, for N easily amounts to several million people.

tional $u_i = \sum_{t=0}^T \delta^t u_{s_i}(q_t)$, where δ is the (fixed) discount factor, and $u_{s_i}(\cdot)$

is the utility received by the player F in the stage-game s , which depends on the stage profile q_t . Every stage-game Γ_s can then be viewed as a triple $\{N + 1, Q, u_{sF}\}$. Unless explicitly stated otherwise, we limit our attention to the utility functions $u_{sF}(\cdot)$ that are continuous and linear in their outcomes, *i.e.* we shall be dealing with risk-neutral (expected value maximizing) players; the generalizations of our results to the case of nonlinear utility functions are straightforward. It is worth emphasizing that players' payoffs depend on the state and the strategies' profile, but not on time: all players receive the same (undiscounted) payoff whenever profile q is played in state s . Accordingly, players' strategies are 'closed-loop' — they depend on the actual state achieved at any moment t , but, in general, not on t alone. Without loss of substance we limit our attention to the case when payoffs of all individuals are the same, while payoffs for the firm differ as a function of the number of individuals who have invested in it. Recalling that $|N| < \infty$ and that the firm undertakes no investment, it immediately follows that the pyramid is doomed to collapse whenever the inflow of new deposits becomes too weak to cover the increasing debt payments. This latter fact also implies another important feature: the Ponzi game is terminating, *i.e.* the last stage T is reached in finite time. This implies that the states Γ_s cannot form infinite cycles: essentially, the game Γ is a (non-recurrent) sequence of stage games which ends with the firm's defection.

6. $p_{s's',t}[\Gamma_{s,t+1}|\Gamma_{s',t}, q_t]$ is the *transition function* defined for $t \geq 1$ that maps any state s' that holds at t to state s at $t + 1$, provided profile q_t was played at t ; and similarly for every t . Consistency of this definition for the game Γ implies that further restrictions are to be imposed on this function, which shall be stated in due course. Sequences of profiles $h_t = \{q_0, q_1, q_2, \dots, q_{t-1}\}$ defined inductively for $t = 1, 2, \dots$ and corresponding to all possible profiles played at stages 0, 1, 2, ... with transitions given by $p(\cdot)$ are called *histories* of plays at t (we usually omit the state index s because set Q is the same for all s). The history $h_0 = \emptyset$ is called *empty history*; the set of all possible histories of length t is $H_t = [Q_s]^t$ for all $t \leq T < \infty$, and the set of all histories of game Γ is

$H = \bigcup_{t=0}^{\infty} H_t$. In stochastic games it is customary to assume that the history

of moves prior to t is "summarized" by the state at period $t - 1$, which is a manifestation of the *Markov property*. With this property, finiteness of S , and exogenously given γ_0 , the evolution of states via $p(\cdot)$ can always

be defined, and H is measurable even in the "classical" sense. It may also be noticed that the set of all sequences $\{h_t\}$ forms a nondecreasing sequence of algebras for every t .

Several possible evolution scenarios given by $p(\cdot)$ can be considered. The first one is *deterministic*, with transition probabilities degenerate for all s' , *i.e.*

$$\begin{aligned} \rho_{s's, t} [\Gamma_{s, t+1} | \Gamma_{s', t}, q_t] &= 1, & \text{if state } s \text{ is the immediate successor of} \\ & & \text{state } s', \text{ given that profile } q \text{ is played} \\ & & \text{at } t; \\ \rho_{s's, t} [\Gamma_{s, t+1} | \Gamma_{s', t}, q_t] &= 0 & \text{otherwise.} \end{aligned}$$

In this case, the evolution of states is given by a *deterministic* function of the current state and strategies profile, which takes place in the complete information case. If information is incomplete, *stochastic* evolution with nondegenerate ρ will be needed, where each state-profile combination may have more than one immediate successor. Finally, in both deterministic and stochastic cases we may consider *evolutions with noise* that are conceptually different. There, the transition function assumes the form $\rho_{s's, t} [\Gamma_{s, t+1} | \Gamma_{s', t}, q_t, z_t]$ where the last variable z_t is a stochastic term, interpreted as unplanned and unpredictable changes, or trembles of any player 1, 2, ..., N , $N+1$ about the component $q_{\pi-1}$ of his or her behavioural strategy, but within the same strategies space. Distinction between strategy and tremble is thus qualitative in character; to capture it properly, a player's utility function ought be redefined as $u_{sF}(q_t, z_t)$ for dynamics with noise. Additional (measurability) issues would naturally arise in these stochastic settings; however, throughout this paper we do not need to invoke disturbance terms, working instead with its deterministic approximations.⁸

Omitting the disturbance term, the order of moves in a stochastic Ponzi game of complete information for $t = 0, 1, 2, \dots$ would be as follows:

1. Independent of each other, players select their actions, thereby determining a profile q_t , observed by all players.
2. Payoffs $u_{sF}(q_t)$ are incurred and added to the utility functional of every player;
3. The next state is determined according to the function $\rho_{s's, t}(\cdot)$.

⁸ Moreover, we believe that an explicit consideration of stochastic components, while bringing unnecessary complications, would divert attention from more substantial issues pertinent to the Ponzi games.

Total payoffs for all players linearly depend on the state s , the action profile q , and the transition function $p(\cdot)$.

The above description is standard for any finite stochastic game, where equilibrium existence follows from the standard application of the Brouwer fixed point theorem (Sobel, 1971; see also Friedman, 1989) to the sets of players' *stationary policies* $\psi = \{\psi_1, \dots, \psi_N, \psi_{N+1}\}$. Such policies are nothing more than the predetermined collections of behavioural strategies for every Γ_s , one for each player. Sufficiency of analysis in terms of such policies follows from the results of Blackwell (1965) and Denardo (1967) which allow us to apply the standard Brouwer fixed point theorem to the tuple of stage-additive and continuous utility functionals of all players on a nonempty compact and convex subset of the n -dimensional Euclidean space. Other approaches to the proof of Markov equilibrium existence in more general settings (including infinite time horizon and uncountable actions and state spaces) were implemented by Reidel (1979), Parthasarathy (1982), Nowak (1985), Parthasarathy and Sinha (1989), Mertens and Parthasarathy (1991), Duffie *et al.* (1994) and Chakribarti (1999), among others.

These last specifications are not exactly appropriate for the case of Ponzi games. If the individuals knew that the game $\Gamma_s = \gamma_T$ is to be played next, they all would withhold at that stage, and thus the firm would be better off stopping at $T - 1$. Reasoning backward in the same way, we easily see that if the information is complete and players behave optimally, the pyramid would not grow at all, and the only (sub-game-perfect) equilibrium profile would have been W for all individuals (D for the firm) for all Γ_s . But the pyramids did actually grow; and the question naturally arises whether this might have resulted from rational individual decisions under incomplete information, in contrast to the assumption of complete information as stated above in this paragraph.

One of the results of this paper is that the above argument carries over to the case of *asymmetric* and *incomplete* information, where the firm knows which game Γ_s is played now (*i.e.* it knows its own current payoff), while none of the individuals do. Such a situation corresponds with the actual state of affairs, and also characterises the phenomenon of financial pyramids as resulting from boundedly rational (Simon, 1978) behaviour. We strongly maintain that bounded rationality is an appropriate assumption, not least because game Γ is characterized by nontrivial information flows, which are potentially confusing even for a player who is otherwise (substantively) rational.

Intuitively, the problem of optimal strategy selection under incomplete information can be characterized as follows. A firm's defection means it

is no longer able (or willing) to stay in business; therefore, it happens at stage T . By contrast, because the population of individuals is large, it is natural to suppose that no withdrawal of any single individual alone can force the game to end. In other words, the firm's strategy is not conditional upon a single opponent's strategy, but upon that of a population; it is the collective behaviour of these individuals that matters to the firm. Putting it otherwise, instead of making an assumption (extremely heroic, in our view) that the firm explicitly conditions its behaviour upon the strategy of each and every of its N opponents, — we suggest that the firm's strategy is based on the anticipation of the aggregate behaviour of the population of individuals. This decision strategy is certainly feasible and presumably has actually been used by Ponzi firms. In terms used in optimal control literature, the firm has an imperfect state information, that is, it conditions its decision on *sufficient statistics*. These statistics constitute a family of real-valued functions defined on the set of all histories which, when used to describe the state, entail no loss of expected utility as compared to the exact description of the state. An example of such a sufficient statistic for the firm is the aggregate history of investments available at t ,

$$h_{N+1,t} = \{n_0, n_1, n_2, \dots, n_{t-1}; q_{N+1,0}, q_{N+1,1}, q_{N+1,2}, \dots, q_{N+1,t-1}\},$$

i.e. a sequence of numbers n_t of I-strategists at each time period, as these are observed by the firm, together with the firm's own past strategies.⁹ Henceforth we suppose that such observations are correct at any time.

The case of individuals is different in two respects. First, while observing the full history of the firm's past plays, no individual can observe the firm's payoffs at any stage of the game (violation of this requirement would endow the individual with insider information, in which case he or she is hardly different from the Ponzi firm manager). Second, individuals make their decisions independently of each other. It follows that in a simultaneous-move stage game, no individual is certain which of the games Γ_s is played at stage t . Under these conditions, the individual's task is to select an optimal strategy for each $t = 0, 1, 2, \dots, T$, given his or her expectations about profile q_t selected by the population of individuals¹⁰ at t ; and his or her probabilistic beliefs about the firm's type. In

⁹ Use of similar notation — h_t for history and h_t for information of player F at t , — should not be confusing, but stresses the generic unity of both notions.

¹⁰ Strictly speaking, a population profile depends on the strategy of every individual; however, an individual's contribution is negligible in a sufficiently large population. This simplification (customary in population games) is used henceforth without special notice.

terms used in optimal control literature, individuals also have to make a rational decision under imperfect state information, which may be represented by the (observable) information vector $h_t = \{q_{t0}, q_{t1}, q_{t2}, \dots, q_{t-1}; q_{N+1,0}, q_{N+1,1}, q_{N+1,2}, \dots, q_{N+1,t-1}; \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_{t-1}\}$. Here, all q_t and all q_{N+1} are, respectively, this individual and the firm's moves in periods 0, 1, ..., $t-1$, and ζ is a (possibly null) vector of other relevant observations for the same periods, such as the strategies of a sample of other individuals in games $\gamma_0 \dots \gamma_{t-1}$. This vector h_t plays the role of an individual's sufficient statistic which, symptomatically, need not coincide with the sufficient statistic used by the firm. It follows that individuals are ignorant not only about Γ_s to be played at t , but also about the firm's beliefs at that stage, which can be estimated at most probabilistically.

Equipped with these definitions, we may complete the description of the Ponzi game under incomplete information by introducing its last component:

7. $\mu = \{\mu_t\}$ is the set of players' *probability measures* over the state space S (physical uncertainty) and other payoff-relevant parameters of the game, such as players' beliefs about the strategies of other players. Although S itself is finite, this measure is not trivial because of the belief component, which shall be explicitly discussed in Section 4. Clearly, since the firm knows perfectly its payoffs at every t , the marginal distribution of $\mu_{N+1,t}$ on the state space S is degenerate for all t — $\mu_{N+1,t}(s_t | h_{N+1,t}) = 1$ if s is the game to be played at t , 0 otherwise. By contrast, every individual can at most probabilistically specify the game Γ_s to be played at any t (as well as the beliefs and strategies of other N players at that stage). Accordingly, marginal measures μ_t on S are non-degenerate, $\mu_t(s_t | h_t) < 1$, $\forall s$ and $\sum_s \mu_{t,t}(s_t | h_t) = 1$.

This definition implies that the stochastic Ponzi game is a game of perfect recall and of incomplete information. The timing of this game, beginning from γ_0 , is as follows:

1. Nature's initial move determines the firm's possible payoffs for the initial stage-game γ_0 ;
2. Nature reveals the true initial state to the firm but not to the individuals;
3. The firm, on the basis of its sufficient statistics, chooses the strategy for the current period;
4. Individuals, without observing the firm's type or its last move, choose an optimal strategy given their beliefs and conditional upon

their information vectors. The resulting profile is observed by all players.

5. Payoffs $u_{sF}(q_t)$ are incurred, discounted and added to the utility functional of every player;
6. The next state is determined according to the function $\rho_{s's, t}(\cdot)$ and is revealed to the firm but not to individuals.
7. Stages 3–6 above are repeated for all subsequent periods up to T .

Game-tree representation of such a game is prohibitively complicated. To get some idea of its complexity, a simplified extensive-form representation for a single stage game Γ_s is given in Fig. 1 for a particular case of complete but imperfect information, where none of the players observes the moves of the others while making their own. The multi-stage variant of this tree slightly resembles the centipede game: a growing pyramid will evolve in the next period unless one player (the firm) drops by defecting.

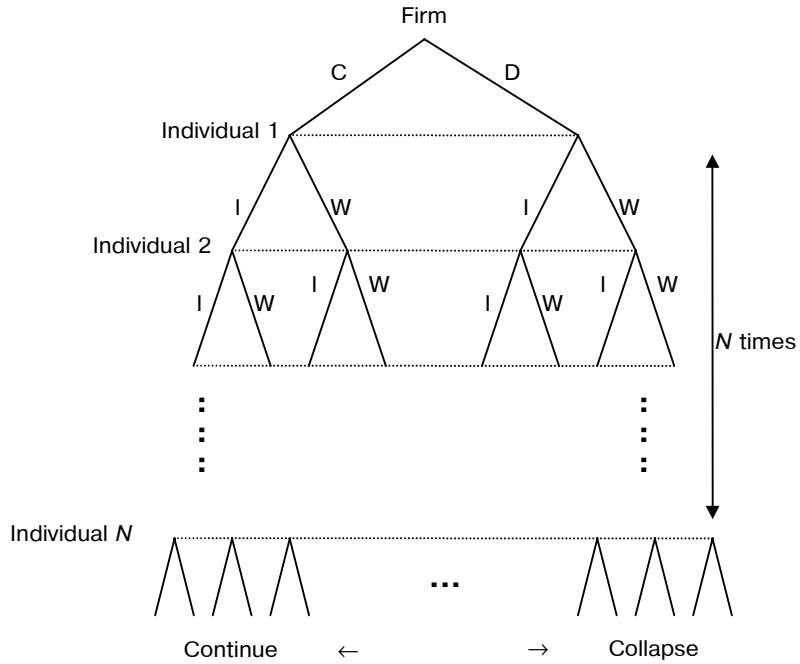


Fig. 1. Stage-game representation in extensive form (complete but imperfect information).

The incomplete and imperfect information case is more complicated; for this case, we are unaware of any general result of equilibrium existence. Fortunately, the specific case of the Ponzi game makes possible its complete equilibrium analysis, which shall be provided in Section 4. Before moving on to this analysis, we want to stress again the importance of the independent decision-making, which is made explicit by Fig. 1. This fact allows us to think of every stage-game Γ_s as of N separate games played at every stage, where a single firm plays the same strategy in every "board" against N independently acting individuals, and each of these N games, in turn, may be thought of as a two-player simultaneous-move game of the firm against a single individual. This last perspective is incomplete but is of some importance for it might well appear appropriate to some investors.

3. PONZI GAME: PARAMETRIC EXAMPLE AND STRATEGY OF THE FIRM

In this section we complete our description of the Ponzi game by formalizing the verbal considerations and assumptions made above. Consider a game involving a population of individuals (actual and potential investors, who may or may not know the firm's intentions) against a single firm — an intended builder of a financial pyramid. This characteristic means that the firm's aim is to accumulate as much money as it can and escape with this money at the moment in time when this amount will be at its maximum (with some abuse of notation, we call this accumulated amount *capital*, denoted by k_t). The Ponzi game Γ is a sequence of stage-games, and in each of these, the Ponzi firm, which must obviously start by Cooperating, has two strategic choices. First, it can continue to cooperate (C) with the public (by paying the promised interest rate on every current investor's deposits and returning deposits to those who want to Withhold). Alternatively, it can Defect (D), in which case it pays no interest, and escapes with the debtholders' cash deposits. Since D is tantamount to the end of the game, this choice of strategy means that the firm escapes with all its current capital short of a fixed bequest B . The amount B may be understood as the value of its fixed assets and/or real option for future growth, which is very likely to be low anyway, especially in an economy characterized by poor prospects for normal business activities.

In turn, individuals acting independently can either invest in the firm or withhold from investing. These decisions are made anew at every stage of the game. The shares of individuals playing I or W or, equivalently,

probabilities that a randomly selected individual will play these strategies at any t , are denoted by q_I and $q_W = (1 - q_I)$, respectively. As argued, any defection on part of the firm creates strong disincentives to invest. It is common knowledge that once the firm defects, it will never return to fair business activities. We shall refer to this property as *responsiveness*. In contrast, the decision to invest may be caused by different factors, the most obvious of which are interest rates. In the Russian pyramids of 1994, these were atypically high, easily reaching 20 to 40% per month, given that the "safe banks" or market rates hardly exceeded 5 to 10%, and the monthly rate of inflation amounted to some 20%. Thus, placement of one's savings in the highly rewarding firm might have been a way to save one's money, and even improve one's well-being in real terms. This strategy is certainly *rationalizeable* in the game-theoretic sense, but also from the viewpoint of any potential investor free of *money illusion*. Since real savings were not the only motive for investing in the pyramid, by simplifying matters, we commonly set the discount rate to 1. We normalize the market interest rates to 0, and use interest rate d , which is fixed throughout the pyramid's existence and is a strictly positive real number equal to the difference between the value promised by the pyramid and the market interest rate. (Of course $d < 1$, and perhaps significantly so, for otherwise potential investors will not take the offer seriously at the very outset.) Finally, for technical reasons the population of individuals is assumed to be sufficiently large to appeal to the Law of Large Numbers (LLN). This in particular allows us to represent individual deposits by their average amount, denoted by M . To obtain some of our results below, we also need to assume that some subgroups of individuals are also sufficiently numerous to appeal to the LLN.

Of several possible formulations we stick to the definition of payoffs in terms of changes from the initial wealth position. This formulation is supported by most psychological findings (Kahneman and Tversky, 1979). Accordingly, for every period t , the payoff to every withholder is 0 (we assume away the existence of other profitable investment opportunities). Investors receive Md if the firm cooperates, and $-M$ if it defects and escapes (short of bankruptcy procedures, investors fail to retrieve any share of bequest B). Each individual who invests at t adds M to k_{t-1} — the value of firm's capital at the end of the previous period. For $t = 1$, the amount $k_0 > 0$ corresponds to the firm's own capital endowment). Let n_{t-1} denote the number of the firm's investors (I-strategists) at the beginning of period t . Those who decide to invest at t , yet were withholders before, constitute an inflow of I-strategists, Δn_t^+ . Those who withhold at t , yet were investors at $t - 1$, constitute an outflow of Δn_t^- . The net inflow

of I-strategists at t is then given by $\Delta n_t = \Delta n_t^+ - \Delta n_t^-$, and the deposit increment is $\Delta k_t = \Delta n_t M$, which number may be positive or negative depending on the relative shares of individual investors that join or quit the pyramid. Total capital at the end of period t is given by $k_{t-1} + \Delta k_t$ minus interest payments (if the firm cooperates) or by bequests (if the firm defects). For simplicity we suppose that all decisions and payments are made within a single period; then, the exact number of individuals to whom the firm owes the promised interest at period t is $n_t = n_{t-1} + \Delta n_t$, and the firm's payment due amounts to $n_t M d$. The explicit characteristic of the dynamics of n_t is of major importance and shall be discussed in Section 5.

The payoff-relevant information for every stage-game may be summarized as a 2×2 bimatrix game of a single individual against the firm (Table 1). Since the firm actually plays against the population, but not against separate individuals, such a game shall be referred to as *truncated*; it shall be shown in the sequel that this representation is also not deprived of meaning.

Table 1. A "single player vs. firm" truncated game.

| Firm | Individual | Invest | Withhold (W) |
|-----------|------------|---|---|
| Cooperate | | $k_{t-1} + \Delta n_t M - n_t M d, M d$ | $k_{t-1} + (\Delta n_t - 1) M - (n_t - 1) M d, 0$ |
| Defect | | $k_{t-1} + \Delta n_t M - B, -M$ | $k_{t-1} + (\Delta n_t - 1) M - B, 0$ |

In case of individual withholding, unity is subtracted from both the capital increment and interest payment, because that choice of a single individual subtracts M from the firm's capital and saves $M d < M$ to its due payments. And of course, one must not forget that in every time period, the firm plays not one but N games as in Table 1 (see Fig. 1), and its payoffs are conditional on the profiles in each of these games. The net inflow of investors Δn_t thus completely determines the firm's payoff at the next stage. Accordingly, the dynamics of Δn_t uniquely determines the sequence of transitions $\rho(\cdot)$ at every t . We emphasize again that the firm cannot and need not base its decision on the psychology of the marginal guy: population dynamics alone matter.

Up to stage t , the number of investors is observable by the firm, but even though the state space is finite, it is not obvious that the firm's probability over possible values of n_t is well-defined in the case of incomplete information. We show that this is the case in the next section.

By now we observe that since $N < \infty$, the space of n_t 's is trivially measurable for all t , so that the probability measure over possible values of n_t at $t - 1$ can always be defined for the firm. Under these circumstances, the payoff structure as specified above suffices to derive the rational strategy for the capital-maximizing Ponzi firm that manages to build a pyramid, *i.e.* to attract an amount of money that exceeds k_0 (which is, of course, the only case of interest). Namely, the firm should cooperate as long as it expects the inflow of capital to exceed its outflow:

$$E_{n(t)}(\Delta n_t^+)M \geq E_{n(t)}(n_t d + \Delta n_t^-)M, \quad (1)$$

where expectation is taken over by the values of $n(t)$ — the number of I-strategists at t estimated at the end of period $t - 1$.¹¹ If (1) is violated, the firm's decision will in general depend upon the relationship between B and $E_{n(t)}(n_t d + \Delta n_t^-)M$. If $B < E_{n(t)}(n_t d + \Delta n_t^-)M$, the opportunity cost of defection is less than the payments due, and the firm will play D, whereas if $B \geq E_{n(t)}(n_t d + \Delta n_t^-)M$, the current cost of operation is below sunk costs, and it may be worth remaining in business.¹² This latter case is somewhat similar to the instance when $k_t \leq k_0$, $\forall t$; in both cases the firm foregoes more than it gains. In such cases, we say that the pyramid has been *unsuccessful*, and we shall assume them away, sticking to the case when $B < E_{n(t)}(n_t d + \Delta n_t^-)M$.

Collecting our assumptions, we easily obtain the following almost intuitive result:

Proposition 1. Under the *responsiveness* condition, a *successful* pyramid's optimal strategy consists of constantly cooperating whenever (1) holds, and defecting once and forever when it expects (1) to be violated.

For the sake of completeness, we address yet another possibility: the firm may want to continue cooperation if it expects (1) to be violated because it believes that the inflow of deposits has been only temporarily weakened at period t , and its capital will be more than recovered in the near future. In such cases, we shall say that the capital dynamics have been non-monotonic, which we may also assume away. Under this condition, the above proposition may be refined to the following:

¹¹ Note in passing that since the stage-game is a simultaneous-move, the firm must decide about defection at T in the previous period, $T - 1$. This may be thought as an advanced collection of assets prior to stealing them.

¹² This is, of course, a version of the standard shutdown condition in the theory of the firm.

Proposition 1a. Under the *responsiveness* condition, whenever a *successful* pyramid observes a *monotonically increasing* inflow of deposits, it should constantly cooperate; whenever it expects (1) to be violated for the first time, it should defect once and forever.

The proof of both propositions is constructed as a solution to the ensuing stochastic optimization problem for the optimal stopping time (e.g., Kushner, 1971, or Bertsekas, 1976), and is provided in the Appendix. As follows from that proof, the optimal dynamic programming algorithm for a successful pyramid is as follows:

Start playing C and for every stage $t = 1, 2, \dots$, compare

$$\begin{aligned} & \max E_{n(t)}\{k_{t-1} + \Delta n_t M - n_t M d\} + V_{t+1}[k_t + \Delta n_{t+1} M - n_{t+1} M d], \\ & \max E_{n(t)}\{k_{t-1} + \Delta n_t M - n_t M d\} - [\Delta n_{t+1} M - n_{t+1} M d]; \end{aligned} \quad (2)$$

when the second of these expression exceeds the first one, stop the pyramid by playing D at $t + 1 = T$, otherwise continue as before.

The solution of problem (2) determines a unique best-reply correspondence for the firm in terms of payoff-relevant histories, *i.e.* without explicit appeal to the states' formulation of stochastic games. It is not difficult to see, however, that there exists a substantial morphism between the two formulations. This is because any payoff-relevant history at any moment t corresponds to a unique stage-game $\gamma_t = \Gamma_s$ in terms of the firms' payoffs. Accordingly, the firm's strategy also determines the optimal stationary policy of the firm in the stochastic game Γ . Proposition 1 shows that from (2), such a strategy consists of a sequence of pure strategies $\{C \text{ in } t = 1, 2, \dots, T-1, D \text{ in } T\}$, where T is free; generally, it cannot be known *a priori*. The optimal stopping time thus depends on the firm's expectations about the dynamics of n_t , to which we are moving now.

4. INDIVIDUAL STRATEGIES AND EQUILIBRIUM ANALYSIS

A. Two kinds of individuals

Individual behaviour is perhaps the most puzzling problem of modern Ponzi games, for their behaviour can hardly be treated as rational. One possibility would, of course, be to say that individual participants of the pyramids were *irrational* subjects of transitional economies, essentially different from the rational subjects in the rest of the world. This viewpoint has its proponents in both Western (Gaddy and Ickes, 1998) and East European (Abalkin, 1997) countries; and may be backed by the observed

differences in individual behaviour in some specific contexts (Roth *et al.* 1991; Slonim and Roth 1998; Cameron 1999). Nevertheless, we are unaware of any convincing evidence which might suggest that people with different national, cultural, historical backgrounds tend to be *systematically* more or less rational in their behaviour under risk, especially when their motivation is salient enough. More specifically, a growing body of experimental evidence implies that people's behaviour toward risk tends to be uniform throughout the world (Kachelmeier and Shehata 1992; Belianin 1999), which clearly speaks in favour of the opposing view. Therefore, we think that a proper explanation for individual involvement in the Ponzi schemes is to be sought elsewhere, and in particular, in different perceptions of economic institutions and related beliefs.

Characterizing these beliefs with respect to financial pyramids, one should immediately realize that the Ponzi game is asymmetric. The optimum strategy of every individual depends on the strategy of the firm; whereas the optimum strategy of the firm does not depend on individuals' behaviour, but only on the aggregate behaviour of the population of individuals. Thus many quasi-rational individuals might have tended to "simplify" the problem by reducing it to the game in Table 1, and sought for an equilibrium profile in this repeated truncated game. Finally, some talks to the Russian participants of the Ponzi games suggests that a substantial part of individual investors simply failed to understand they are playing against thieves, being instead naive enough to believe every word of the firm and really trusting in its ability to generate returns above the market rate. In such cases, individuals cannot be assumed to apply any probabilistic reasoning to assess the likelihood of the pyramid's collapse.

To take into account this diversity of approaches, beliefs and strategies, we partition the entire population into two types: *naive* and *sophisticated*, indexed when necessary by superscripts n and s , respectively. Naive individuals were *boundedly rational* and 'trusting': until the pyramid had collapsed, they uniformly believed with probability 1 that the Ponzi firm is generally honest, or at least hardly more risky than any other bank. Their inflow into the pyramid depends mainly upon information about the existence of such investment opportunity and the firm's reputation as a contract-fulfilling institution. These factors are conveniently captured by the mechanism of strategy selection, which is based on comparison of one's own current payoffs to those of other subjects, resulting in the population's gradual drift towards a more profitable strategy. In other words, alongside the pyramid's growth, more and more people may invest just because they know others who already did and had been rewarded. This

last argument naturally connects their *bounded rationality* to the framework of evolutionary game theory, as discussed in detail in the next section.

By contrast, sophisticated investors acting independently are *substantively rational* in the sense of Simon (1955; 1978) they maximize expected utility, given their current beliefs about their opponent's strategy, and these beliefs are updated according to Bayes rule. Unlike naive investors, they have heterogeneous beliefs about the firm's defection which, in addition, varies with time as captured by the appropriate probability measure. In other words, these people realize that an investment in such a firm is very hazardous, but they have an even stronger belief that they are smarter than the firm and will withdraw before the scheme collapses.

B. Strategies of sophisticated individuals in a dynamic game

Under the settings specified above, the equilibrium analysis of the Ponzi game makes sense only for substantively rational players: sophisticated individuals and the firm. When developing this analysis, it is useful to extend the truncated Ponzi game of the pyramid against a single individual (Table 1) to take into account incomplete information. In this case, a single individual decides whether or not to invest in the pyramid at any given stage t , but the firm's payoffs and actions are conditional upon the firm's possible types. In the present context, such types correspond to possible values of the firm's capital k_{t-1} and its expected net inflow of investors at period t , both of which are known to the firm but not to the individuals who have only a well-defined probability measure over these states. Index all (possibly infinitely many) types by $\theta_1, \theta_2, \dots, \theta_\theta$, and without loss of generality order these types according to their indices from worst to best. Then, there is a type θ_t^* so that, for any type $\theta_t^+ > \theta_t^*$, prospects for the pyramid's growth are positive, and the firm wants to cooperate; at any $\theta_t^- \leq \theta_t^*$, it has to defect (we suppose that indifference is resolved in favour of defection). The resulting truncated stage-game of incomplete information is presented in Table 2 (superscripts denote values of k_{t-1} , Δn_t and n_t specific to all possible firms' types).

Since no single individual's contribution can alter the firm's decision to cooperate or defect, we immediately see that either the upper (C) or lower (D) half of firm's strategies are dominant, conditional upon (1) or, equivalently, upon whether $\theta_t^+ > \theta_t^*$ or $\theta_t^- \leq \theta_t^*$, respectively. Therefore,

Table 2. Truncated stage-game of incomplete information.

| Individual Firm | Invest | Invest |
|----------------------|--|--|
| 1, Cooperate | $k_{t-1}^1 + \Delta n_t^1 M - n_t^1 M d, M d$ | $k_{t-1}^1 + (\Delta n_t^1 - 1) M - (n_t^1 - 1) M d, 0$ |
| 2, Cooperate | $k_{t-1}^2 + \Delta n_t^2 M - n_t^2 M d, M d$ | $k_{t-1}^2 + (\Delta n_t^2 - 1) M - (n_t^2 - 1) M d, 0$ |
| ... | ... | ... |
| Θ , Cooperate | $k_{t-1}^\Theta + \Delta n_t^\Theta M - n_t^\Theta M d, M d$ | $k_{t-1}^\Theta + (\Delta n_t^\Theta - 1) M - (n_t^\Theta - 1) M d, 0$ |
| 1, Defect | $k_{t-1}^1 + \Delta n_t^1 M - B, -M$ | $k_{t-1}^1 + (\Delta n_t^1 - 1) M - B, 0$ |
| 2, Defect | $k_{t-1}^2 + \Delta n_t^2 M - B, -M$ | $k_{t-1}^2 + (\Delta n_t^2 - 1) M - B, 0$ |
| ... | ... | ... |
| Θ , Defect | $k_{t-1}^\Theta + \Delta n_t^\Theta M - B, -M$ | $k_{t-1}^\Theta + (\Delta n_t^\Theta - 1) M - B, 0$ |

this strategic-form game suggests that there are two equilibrium components in pure strategies, conditional upon the firm's type (and none in mixed strategies). An individual who believes that the firm will cooperate, at the stage-game t with probability above $1/(1 + d)$ has to play I; and if the firm is of type $\theta_t^+ > \theta_t^*$, it will continue to cooperate at t , so that any $(\theta_t^+ C, I)$ profile constitutes an equilibrium. Another equilibrium is $(\theta_t^- D, W)$, which occurs when individuals do not want to mess with a risky firm (in fact, it is *risk-dominant* in the sense of Harsanyi and Selten, 1988). Intuitively, the latter equilibrium can be deemed "better" than the former; however, profiles of the (C, I) component cannot be rejected as disequilibrium, not even on the grounds of known equilibrium refinements. The following formal result proven in the Appendix stems from the fact that if the firm's intention to defect in the stage-truncated game at t is not common knowledge, individuals may have a range of consistent beliefs about the firm's strategy, some of which may be true.

Proposition 2. In the strategic-form truncated stage game in Table 2 (as part of the Ponzi game), equilibrium components $(\theta_t^+ C, I)$ and $(\theta_t^- D, W)$ will contain *perfect, sequential, proper, stable* and *essential* equilibria whenever the firm considers it optimal to cooperate and defect, respectively, and the individual attaches sufficiently high probability to these events.

This proposition seemingly justifies a possible extension of the $(\theta_t^+ C, I)$ profile program to the full Ponzi game, as long as an individual's uncertainty about the likelihood of defection at t is captured in a "proper" manner. Arguments of that sort might be (and probably actually were) advanced by some sophisticated individuals to back the following *cautious* strategy: *Invest at early stage, but be 'sufficiently prudent', and withhold before the expected collapse of the pyramid.* In other words, a sophisticated investor with a sufficiently high discount factor tries to solve his own stochastic control problem in a truncated Ponzi game as given by a sequence of stage-games from Table 2.

$$\sum_{\tau} u_{\ell\tau} = \sum_{\tau} \int_{\theta} \delta^{\tau} e_{\bar{R}} u_{\bar{R}}(e_{\bar{R}}, e_{N+1,\tau}, \theta_{\tau}) \mu(d\theta_{\tau}) \quad (3)$$

As long as the firm's intention to defect comes together with the individual's intention to stop, profile (C, I) at every stage prior to T and (D, W) at T in this truncated game would "look like" an equilibrium which shall also be sequential, perfect, subgame perfect and Markov perfect.

However appealing and successful in practice, this argument is wrong for the following reason. The cautious strategy is arguably optimal in an extension of a truncated game; however, when the individual plays against the firm, the firm is not playing against this individual alone, but against the population. But the population's "strategy" is not observable by any single individual; it can be at most probabilistically evaluated via some statistic.¹³ Such probability space is explicitly constructed in the Appendix; assuming this probability space exists, the supposed cautious strategy should maximize not the short-term payoff (3) over some $\tau \subset T$, but payoff

$$\begin{aligned} V_{\ell T}(\psi_{\ell}) = & G^1(1,0) + \delta \sum_{s'} q_{s'1} \left(\sum_s p_{s's,1}(q_{s',1}) G^2(2,1) \right) + \dots + \\ & + \delta^T \sum_{s'} q_{s'T} \left\{ \sum_s p_{s's,T}(q_{s',T}) \dots \left[\sum_{s'} q_{s'T-1} \left(\sum_s p_{s's,T-1}(q_{s',T-1}) G^T(T,T-1) \right) \right] \right\}, \quad (4) \end{aligned}$$

where $G^n(n,n-1)$ is defined inductively as $\int_{\theta} q_{\ell n} u_{s\ell}(q_n, \theta) \mu(d\theta, s | h_{\ell n-1})$ for all $n = 1, 2, \dots, T$. In equation (4), every $q_{\ell t}(h_t)$ is an $H_t \times S$ — measurable

¹³ It is worth mentioning that the calculation of this probability, if possible, is far from being trivial!

function whose range is a probability distribution over one and the same set of actions at t , namely, $\{I, W\}$. Policies ψ_F which maximize this payoff, constitute the true best response in the Ponzi game. Is it possible to find some nontrivial profile compatible with the *cautious* strategy? A little reasoning suggests the negative answer to that question; however, before reaching this conclusion we need first to show that the mere maximization problem in (4) is appropriately formulated. This formulation is provided in the next subsection.

We conclude the present discussion by noticing that those individuals who were successful in playing the cautious strategy were in fact not-so-wise or virtuous. Not wise, because the cautious solution of a dynamic optimization problem for optimal stopping time has to be found by maximizing *expected* gain; it does not provide an efficient algorithm to locate the last period of the pyramid's existence. Accordingly, those who succeeded in playing this strategy did so only by *luck*, unless they were insiders of the Ponzi firm, in which case they are hardly distinguishable from its managers. And they were not virtuous because their wealth increased at the expense of other fellow investors who happened to invest later. In that sense, these cautious individuals themselves played the role of "little Ponzis"!

C. Equilibrium analysis of the Ponzi game

The considerations provided above led to the following proposition, which is stated in its full generality:

Proposition 3. The only equilibrium of the Ponzi game Γ under incomplete information and the above specifications is a trivial one: (D, W) at every stage.

Proof of that proposition requires some further elaboration, beginning with a definition of the equilibrium concept for the stochastic Ponzi game Γ of incomplete information. It is formulated below in standard (Bayesian) terms, as the mutual best response of players' behavioural strategies conditional upon their respective types. For every history h_t , $t = 0, 1, 2, \dots$ let $\mu_{Ft}(\theta_{-F}|\theta_F, h_t)$ be the (joint) subjective probability of player F at t that his N opponents at t are of types θ_{-F} given that he is of type θ_F himself and given h_t , his or her information vector at t .

Let $v_{Ft}(\psi_F) = \sum_{\tau=t+1}^T \delta^\tau E[u_F(q_{F\tau}, \theta, s, h_{F\tau})]$ be player F 's expected continuation payoff for the stages that follow t if he adopts policy ψ_F . The set of *stationary policies* $\{\psi_F\}$ for every player F has a very simple structure; since

mixed strategies are ruled out, it is a $|2^S|$ -set of all combinations of I and W (for individuals) or C and D (for the firm). Therefore, $v_{\bar{t}}(\psi_{\bar{t}})$ for every $\psi_{\bar{t}}$ is defined as long as both subjective probabilities $\mu_{\bar{t}}(\theta_{-\bar{t}} | \theta_{\bar{t}}, h_{\bar{t}})$ are properly defined for all $\tau = t + 1, t + 2, \dots, T$, and transition probabilities $\rho_{S', t}[\Gamma_{S', t+1} | \Gamma_{S', t}, q_t]$ are properly defined for all t . This last condition is again easily met in the Ponzi game. To every possible value of k_0 (initial capital of the firm) corresponds a finite set of cardinality 2^N whose points are possible profiles q_t . Because this last set is trivially measurable, and the game is over after a finite number of steps, the transition probability is uniquely defined for every k_0 and history h_t , and game $\Gamma_{S', t+1}$ is uniquely determined by $\Gamma_{S', t}$ and profile q_t played at stage t .

For every player F and every $t = 0, 1, \dots$ pure strategies are $e_{\bar{t}} = \{I, W\}$ if $F = 1, 2, \dots, N$, and $\{C, D\}$ if $F = N + 1$. For respective cases, let $q_{-\bar{t}}$ be the profile of other players' strategies. Equilibrium policy $\psi_{\bar{t}}$ is a collection of $e_{\bar{t}}$'s, one for every stage-game, which for every player of type θ_F maximizes

$$\begin{aligned} V_{\bar{t}}(\psi_{\bar{t}}) = & \int_{\theta_{-\bar{t}}} u_{\bar{t}}[e_{\bar{t}}, q_{-\bar{t}}, s | \theta, h_{\bar{t}}] \mu_{\bar{t}}(d\theta_{-\bar{t}} | s | \theta_{\bar{t}}, h_{\bar{t}}) + \\ & + \sum_{S'} \rho_{S', t} \mu_{\bar{t}}(\theta_{-\bar{t}} | s | \theta_{\bar{t}}, h_{\bar{t}}) v_{\bar{t}}(\psi_{\bar{t}}), \end{aligned} \quad (5)$$

and an equilibrium payoff is the maximized sum of stage utilities,

$$V_{\bar{t}}(\psi_{\bar{t}}) = \sum_{t=0}^T V_{\bar{t}t}.$$

To establish that an equilibrium is (non)existent, it remains to show that definition (5) is consistent by showing the existence of players' types. Following Harsanyi (1967), every player in the game Γ is characterized by a vector "representing certain physical, social and psychological attributes of player F himself in that it summarizes some crucial parameters of player F 's own payoff function u_F as well as the main parameters of his beliefs about his social and physical environment. Each player is assumed to know his own actual type but to be in general ignorant about the other players' actual types." (Harsanyi, 1967, p.171) In other words, player's type θ_F should incorporate his or her beliefs not only about possible state $s \in S$, but also about the types of all players. For static games of incomplete information such space has been constructed by Mertens and Zamir (1985) and Brandenburger and Dekel (1993) who interpreted

each Θ_F in the probabilistic (Bayesian) sense, which in particular implies that every player knows all her opponents to be Bayesians, and to which we adhere, too.¹⁴ The case of the dynamic game is more complicated because players' uncertainty extends over future states, that is, over the continuation game, and players may well disagree in their beliefs about this continuation game (by having different beliefs as to which game Γ_s is played at stages $t, t + 1, \dots$). It turns out, however, that the analysis is greatly simplified by the ability to observe the firm's move, and we shall construct a sequence of universal belief spaces which are obtainable for every stage $t + 1$ from the analogous space at t in a natural way. Resulting spaces shall still be "rich enough" to include all possible future paths and all future beliefs of all players. These beliefs are adjustable in a Bayesian manner, and represent players' types. For every stage t , such types are joint probability distributions on the following sets and subject to following condition:

- 1) set B_t^0 represents "physical uncertainty" about the payoffs¹⁵ of all players in the stage-game t and its continuation game;
- 2) set $\{\Theta_{-Ft}\}$ of all possible beliefs of all other players about B_t^0 and F 's beliefs at t ;
- 3) player F knows (assigns probability 1) his own actual beliefs; and
- 4) conditional upon his or her information, these beliefs are updated according to the Bayes rule at every stage t .

The last two are consistency conditions, while the first two define the *universal belief space* $\Omega_t = B_t^0 \times (\Theta_{-Ft})^{N+1}$ with typical element ω_t , where every Θ_{-Ft} is the set of all probability distributions on $B_t^0 \times (\Theta_{-Ft})^N$ (Cartesian product of B_t^0) and the space of beliefs of other players.

Proposition 4. There exists a well-defined universal belief space for stochastic Ponzi game Γ under incomplete information.

Proof of this proposition, together with the construction of the universal belief space is contained in the Appendix. This proof demonstrates, *inter alia*, that all stage-games play a twofold role in the information structure

¹⁴ More general belief spaces may be considered as well (Epstein and Wang, 1996), but these shall not be considered here.

¹⁵ Generally, physical uncertainty should also encompass the set of possible strategies; however, in the Ponzi game, these are simple and commonly known.

of game Γ . On the one hand, they remove uncertainty about each B_t^0 in terms of the firm's move; on the other, they provide partial information about the true state, paving the way to Bayesian updating. It is also instructive to note that these hierarchies contain the possible actual paths of some stochastic process $\{n_t\}$ in the probability space $(B^\infty, \sigma(B^\infty))$, a process which also uniquely determines the values of $\{k_t\}$ from any given k_0 .

Returning now to the players' strategies, it is worth noticing that as long as $h_{\bar{t}} \neq h_{N+1, t}$, all individuals and the firm will generally make use of different information; thus, there is no guarantee that the two inferred posterior probabilities will match each other. In other words, we have the next proposition.

Proposition 5. If different individuals use different vectors $\{\zeta_t\}$ as a sampling distribution of the past history of behaviour of each other, then no proper universal belief space can be constructed. A simple proof is again contained in the Appendix.

This proposition further emphasizes our earlier claim that it makes no sense to play strategically or "outguess" the firm by providing a substantial reason for its failure. If individuals will ground their strategies on anything other than commonly known information, the stochastic process of the pyramid's growth may not be measurable. To avoid these difficulties, in what follows we have to assume a simpler sort of sampling, limiting our attention to $h_{\bar{t}} = \{q_{\bar{t}}, q_{N+1, t}\}$ (past history of own plays and those of the firm for the individual) and $h_{N+1, t} = \{n_{\bar{t}}, q_{N+1, t}\}$ (past history of own plays and the sum statistic of the population strategy for the firm).

However, even this specification does not restore nontrivial equilibria; and now we are in a position to provide a simple proof of Proposition 3 by contradiction (An alternative proof is just a bit more tedious, which makes use of the fact that the map of best-reply correspondences of all players onto itself fails to be continuous).

Proof of proposition 3: Consider the set ψ of policies and suppose there exists a profile of equilibrium strategies of the players, together with the probability measures constructed in the proof of Proposition 5. Recall first that game Γ reaches at most S alternative states with $|S| = 2^{N+1}$, any of which are related to the previous state via transition function ρ . It follows that the firm's optimum policy ψ_{N+1} will include defection in finite time. Suppose this happens at stage T ; then those individuals who withhold at T will be better than those who invest at that

stage. It follows that the Nash equilibrium profile at the last stage T (Table 2) is (D, W) , and all individuals are to defect. But if so, then, by Proposition 1 the firm would not be interested in keeping the pyramid until T , where its payoff is certain to fall short of interest paid in period $T - 1$, resulting in a cash outflow. Therefore, it will want to defect at $T - 1$, but by then individuals by the same token will consider withdrawing optimal at $T - 1$, resulting in (D, W) at that stage, too. Continuing iteratively back to period 0, we see that this is the only mutual best response profile for every stage game. Q.E.D. (By backward induction the idea of the above proof is clearly similar to the "lemons" market argument (Akerlof, 1970).

The result of equilibrium nonexistence is intuitively pretty clear; however, the intuition behind it is not very easy to extract because of nontrivial information flows. The complexity of these flows is a potential source of confusion; however, results of this section suggest a firm "policy advice": DO NOT PLAY PONZI GAMES!

But real people did not seem to listen to that rule; thus, their behaviour must have been suboptimal. One reason for this suboptimality has just been discussed — it is the erroneous belief in one's abilities to behave optimally when you cannot. Another reason is suggested by the bounded rationality paradigm, specified in the next section in an evolutionary framework, where we explicitly construct a number of adaptive dynamics compatible with the logic of Ponzi games. Both lines of explanations imply that the actual growth of financial pyramids has been a disequilibrium phenomenon motivated by equilibrium considerations.

5. EVOLUTIONARY DYNAMICS OF INDIVIDUAL BEHAVIOUR

The above analysis was essentially Bayesian in its spirit; thus, it restricts attention to the behaviour of sophisticated individuals. On the other hand, it might make sense to suppose that many individuals were naive and did not update their beliefs in a Bayesian manner (even in most indirect sense), using instead satisficing strategies (Simon, 1978). It is now proper time to proceed with an explicit description of both processes, leading to the crucial component omitted thus far — the dynamics of investors' net inflow.

A. Naive individuals

We call *naive* individuals boundedly rational in the sense introduced in Section 4, but since the use of this notion in the literature has often been

somewhat voluntary and loose, we recapitulate our definition. First, naive individuals are not stupid: they understand that more money is better than less and are ready to undertake some efforts to increase their well-being. Second, they trust the "skilful" modern managers and advertising campaigns, and do not expect the Ponzi firm to defect at all. Then, they are *myopic*: instead of "thinking of" the best reply to the firm's possible strategy, they just compare the present yield on the strategy they are currently playing to that of the competing one, and choose the best on the basis of current observations only. Moreover, in choosing an optimal strategy they are socially oriented rather than self-dependent; selecting their strategies, they readily refer to the experience of neighbours. Finally, in our application it also makes sense to allow the subjects to change their strategy regardless of its current performance; *e.g.*, deposits may be withdrawn from the firm for the purposes of regular or unexpected transactions.

The above ideas may be captured by a broad class of different learning rules which leads to the variants of the *replicator dynamics* (Bjornerstedt and Weibull, 1995; Weibull, 1995, Schlag, 1998, Borgers and Sarin, 1997, Arthur, 1993, Samuelson, 1997). All these models stipulate an increase with time of the share of better-fitted strategies within the population. Under somewhat specific, but still sufficiently general conditions, the time evolution of the share of investors among these may be usefully approximated¹⁶ by the equations of the replicator dynamics family (in discrete time¹⁷), due originally to Taylor and Jonker, 1978 (see also Maynard Smith, 1982):

$$\Delta q_i = q_i[u(e_i, p) - u(q, p)]. \quad (6)$$

where $u(e_i, p)$, as introduced earlier, is the expected payoff of an individual playing pure strategy i against an opponent from a different population characterized by mixed strategy p . Finally, expected payoff of a randomly selected member of the population characterized by mixed strategy q , against an opponent playing mixed strategy p is denoted by $u(q, p)$. We shall also need *symmetric* game notation, where an individual plays against a randomly chosen member of her own (large)

¹⁶ Coming from biological sciences (Maynard Smith, 1982), the replicator dynamics (5) are often thought to be less natural in economic applications (Levine, 1997). However, recent research (Schlag, 1998; Arthur, 1996) has shown that its flavour is more general than it might appear. Our work is along the same vein of research, suggesting an economic application of some generalisations of the dynamics (5).

¹⁷ Here and below, time-dependence indices are omitted when no confusion is likely to arise.

population — in this case, her expected payoff shall be denoted by $u(e_i, q)$, and the average one — $u(q, q)$. The firm, of course, is interested not in the share of I-strategists in the whole population, but in the number of its investors. But formally, these two quantities are isomorphic: the *number* of i -strategists, denoted by n_i , evolves with the *proportion* of population playing strategy i , $q_i = n_i/N$, as long as the population size N remains the same. Thus, in our case the vector of such proportions $q = [q_I, 1 - q_I] = [q_I, q_W]$ may be formally associated with the mixed strategy of the population.

The replicator dynamics might appear to be an improper tool to describe individual rationality, since this dynamics deals only with populations, which certainly do not reason, at least in the sense in which we speak of a single person. Such an objection, however, is not sustainable: "mixed strategy" just means that at any moment in (discrete) time, every individual investor plays one of two pure strategies: invest or withhold, whichever choice is deemed better for that individual. Furthermore, the assignment of individuals to play a particular strategy does not imply that everyone is doomed to play this strategy once and forever. In fact, quite the opposite is the case: individuals do have free will, and thus may (and even should) change their strategies when the course of events and/or their own reasoning persuades them to do so. In fact, the rationalistic background of the replicator dynamics is yet more involved: individual decisions are subject to evolution, belief dynamics, fashion and other perceptions of the relative fitness of alternative behaviour strategies, which, of course, may be very complicated. But if separate individuals in a large population, on average, update their strategies according to some well-defined rules, the evolution of strategists' shares within the entire population may be either exactly described or approximated¹⁸ by members of particular families of difference/differential equations. Additional support for this "micro-aggregation" in a form of replicator dynamics is in the nature of the control problem for the firm: explicit conditioning of the firms' strategy upon the strategy of any single individual becomes prohibitively difficult even when the population is relatively small.

Below we construct a variant of the deterministic replicator dynamics in discrete time (adapted from Weibull, 1995), which is explicitly derived from individual adaptation strategies. Weibull calls it the replicator dynamics via *imitation of successful behaviour*.

To introduce evolutionary dynamics into the strategies of naive individuals, we use a *symmetric auxiliary game*, as presented in Table 3. This

¹⁸ This approximation, *inter alia*, allows us to suppress the effects of possible random disturbances z_t with no loss of substance.

may be thought as a sort of "fictitious" play between every two stages of the main deposit game.

Table 3. The symmetric auxiliary game.

| | I (Prob = q_I) | W(Prob = $1 - q_I$) |
|-----------------------|-------------------|----------------------|
| I (Prob = q_I) | Md, Md | $Md, 0$ |
| W (Prob = $1 - q_I$) | $0, Md$ | $0, 0$ |

In an auxiliary game, all members of a large population are originally assigned to play some particular strategy (I or W), the shares (and thus, numbers) of I- and W-strategists corresponding to a population's mixed strategy in a sequence of stage-games (as in Table 1). Although these strategies are fixed for all N players within every period, they evolve as a result of random "matches" with other players from the same population. Such matches may be thought as meetings in the street, in a cafe or at a party, in short, at any place where individuals may share their "investment strategies", revising their current strategy according to its relative fitness comparative to that of their opponent in an *auxiliary* game. Specifically, we suppose that revisions of a current strategy for every member j of the subpopulation of naive subjects follow a Poisson process with arrival rate r_j , and every individual j switches to the strategy i with the probability π_j^i , $\sum_i \pi_j^i = 1$, $\forall i, j$ (in general, both r and π can vary across

subpopulations). If these Poisson processes are statistically independent for every player, the average per unit time review rate of current j -strategists will have a Poisson distribution with the parameter $r_j q_j$, where q_j denotes the fraction of j -strategists in the entire population at t (population size is normalized). Invoking a continuum approximation of the entire population,¹⁹ this *stochastic* Poisson process for the j -strategists' average may be approximated by a *deterministic* flow where the average switch rate from j to i per unit time is given by $q_j r_j \pi_j^i$: the proportion of j -strategists times the revision rate times the probability of this switch. The (total) inflow to strategy i is $\sum_j q_j r_j \pi_j^i$, whereas the (to-

¹⁹ A crucial point at which to get a meaningful approximation of the mechanism being described is that of a sufficiently large population, which allows us to appeal to LLN, complemented by some further technical qualifications (Borgers and Sarin, 1997; Boylan, 1992, 1995).

tal) outflow of i -strategists is given by $\sum_j q_i r_j \pi_j^i$; a fraction $q_i r_j \pi_j^i$ reviews strategy i and continues playing it. In discrete time, the average inflow to the subpopulation i equals

$$q_{it+1} - q_{it} = q_1 r_1 \pi_1^i + \dots + q_i r_i \pi_i^i + \dots + q_n r_n \pi_n^i - q_i r_i \pi_i^1 - \dots - q_i r_i \pi_i^n = \sum_j q_j r_j \pi_j^i - q_i r_i \sum_j \pi_j^i = \sum_j q_j r_j \pi_j^i - q_i r_i \quad (7)$$

In our case, there are only two strategies, I and W, where $1 - q_I = q_W$, so this process is simply

$$q_{It+1} - q_{It} = q_I r_I \pi_I^I + (1 - q_I) r_W \pi_W^I - q_I r_I, \quad (8)$$

where the first component denotes a fraction of those who were to modify their behaviour, but found current strategy (I) superior to the other (W). The second stands for those new I-strategists, and the last one denotes the outflow of I-strategists to strategy W.

To obtain close-form dynamics, further simplification of (8) is needed. We shall concentrate on a particularly compelling specification, also due to Weibull (1995). It assumes that the revision rates are constant across a population (set $r_i = 1, \forall i$), but the probabilities of switching to the strategy of randomly selected individuals depend on the relative fitness of the two strategies. According to this rule, the current j -strategist samples at random another member of the same population, and switches to strategy i if the player he or she meets was an i -strategist and if his or her perceived utility of strategy i is above his or her perceived utility of the strategy j . Maintaining that $u(\cdot)$ are risk-neutral utilities (payoffs from Table 1), the *perceived* utilities will be denoted by $u(e_i, q) + \varepsilon$ and $u(e_j, q) + \eta$, where $u(\cdot, q)$ are expected gains on alternative strategies, and parameters ε and η are subject-specific random variables with known distribution among the population, which capture particular shapes of individual utility functions.²¹ A switch from j to i will then occur

²⁰ Letting the time increment in (6) go to 0, the differential analogue of the above process can be obtained as

$$\partial q_i / \partial t = \sum_j q_j r_j^0 \pi_j^i - q_i r_i^0 \quad (6a)$$

with an accordingly adjusted review rate, i.e., $r_i^0 = r_i \tau, \tau \rightarrow 0$.

²¹ To ensure accuracy of the following derivation, we need to assume that expected values of strategies I and W are the same for all prospective investors. Here we need the assumption of identical beliefs for all naive individuals.

if $u(e_i, q) + \varepsilon > u(e_j, q) + \eta \Leftrightarrow \eta - \varepsilon < u(e_i, q) - u(e_j, q)$; expression $\eta - \varepsilon$ will also be a random variable, assumed continuously differentiable and with known cdf $F(\cdot)$. Under random sampling across q , the probability that a j -strategist will play strategy i at the next stage will then be given by

$$\pi_i^j = \begin{cases} q_i F[u(e_i, q) - u(e_j, q)], & \text{if } j \neq i \text{ (switch to } i \\ & \text{from different } j \text{);} \\ 1 - \sum_{j \neq i} q_j F[u(e_i, q) - u(e_j, q)], & \text{if } j = i \text{ (remain at } j \text{ with} \\ & \text{complementary probability);} \end{cases} \quad (9)$$

and similarly for all strategies. In particular for the case of Ponzi game, the rule (8) results in

$$\begin{aligned} \pi_W^I &= q_I F[u(e_I, q) - u(e_W, q)], \quad \pi_W^W = 1 - q_I F[u(e_I, q) - u(e_W, q)]; \text{ and} \\ \pi_I^W &= q_W F[u(e_W, q) - u(e_I, q)], \quad \pi_I^I = 1 - q_W F[u(e_W, q) - u(e_I, q)]. \end{aligned} \quad (10)$$

With these functions π and constant $r = 1$, recalling that $q_W = 1 - q_I$, we obtain from (8) the following dynamics for I :

$$\begin{aligned} q_I' &= q_I r_I \pi_I^I + (1 - q_I) r_W \pi_W^I - q_I r_I = \\ &= q_I \{1 - (1 - q_I) F[u(e_W, q) - u(e_I, q)] - 1\} + \\ &+ (1 - q_I) q_I F[u(e_I, q) - u(e_W, q)] = \\ &= q_I (1 - q_I) \{F[u(e_I, q) - u(e_W, q)] - F[u(e_W, q) - u(e_I, q)]\}, \end{aligned} \quad (11)$$

and similarly for W . Payoff-monotonicity of (11) is ensured if $F(\cdot)$ is strictly increasing. A linear approximation of these dynamics is always possible near the steady-state (Weibull, 1995); the replicator approximation may be obtained everywhere on the mixed strategies' simplex if F is the uniform distribution given by $a + b[u(e_i, q) - u(e_j, q)]$, $b > 0$. Under this assumption, a version of the replicator dynamics in discrete time may be obtained from (11) as

$$\begin{aligned} q_I' &= q_I (1 - q_I) \{F[u(e_I, q) - u(e_W, q)] - F[u(e_W, q) - u(e_I, q)]\} = \\ &= q_I (1 - q_I) \{a + b[u(e_I, q) - u(e_W, q)] - \\ &- a - b[u(e_W, q) - u(e_I, q)]\} = q_I \{(1 - q_I) b u(e_I, q) - \\ &- (1 - q_I) b u(e_W, q) - (1 - q_I) b u(e_W, q) + (1 - q_I) b u(e_I, q)\} = \\ &= q_I \{2b u(e_I, q) - 2b u(e_W, q)\} = 2b q_I^n [u(e_I, q^n) - u(e_W, q^n)], \end{aligned} \quad (12)$$

Dynamic (12) is a rescaling of replicator dynamics (6) with factor $2b$.

B. Sophisticated individuals

The model considered in the previous section dealt exclusively with naive individuals' behaviour; but the whole population consists of sophisticated ones as well. As discussed above, their sophistication is of little worth however. To the extent this disequilibrium pattern took place, the dynamics of this subpopulation warrant a separate description. Following our earlier notation, let the whole population N consist of N^n naive and N^s sophisticated individuals. In what follows we shall index these respective fractions by superscripts n and s , respectively, so that the number of sophisticated I-strategists at each time period is n_t^s , that of naive strategists is n_t^n , their sum is n_t , and the total number of withholders at t equals $N - n_t$. Since $N = N^n + N^s$, $q_I = (n^n + n^s)/N$ defines the proportion of I-strategists in N (the whole population, without superscript) at any t , we also have $q_I^s = n^s/N^s$, $q_I^n = n^n/N^n$, and the total share of I-strategists (with $1 - q_I$ as the total share of W-strategists) may be represented as

$$q_I = \frac{q_I^n N^n + q_I^s N^s}{N^n + N^s}.$$

Unlike naive individuals, sophisticated ones are heterogeneous: they have different beliefs as to what the firm's strategy will be in the coming stage game. They are also substantively rational in the sense that their subjective probability of the firm's defection is above zero and further is nondecreasing with the time of the pyramid's existence. It may be noted that for the sophisticated individuals, the Ponzi firm has a strong "negative" reputation, namely the individuals behave strategically against a known-to-become-defective firm and decide to withhold depending on the current values of their information vectors h_t , modifying their beliefs via the Bayes rule.

For reasons of tractability we shall henceforth assume that all individuals observe the same history h_t consisting of the firm's moves and their own strategy, but that their beliefs differ only with respect to the initial belief about the time when the Ponzi firm will ultimately defect. Fixing these evolutions for the rest of the paper, any of its paths will be denoted \bar{h}_t (with an overbar, to distinguish it from histories h_t). Consider an arbitrary sophisticated individual F who in every period needs to estimate the posterior probabilities $\text{Prob}(C_{t+1} | \bar{h}_t)$ and $\text{Prob}(D_{t+1} | \bar{h}_t)$. In two cases, the beliefs of sophisticated and naive individuals coincide: if \bar{h}_t contains information about the firm playing D at any time period in the past, then

$\text{Prob}(D_{t+1} | \bar{h}_t) = 1$ and $\text{Prob}(C_{t+1} | \bar{h}_t) = 0$ for all individuals (this is of course a paraphrase of the responsiveness condition). For any other history, making a (reasonable) assumption that the firm's actions are stage-independent from an individual's viewpoint, for all histories \bar{h}_t involving $t + 1$ cooperations in a row, we have

$$\begin{aligned} \text{Prob}(C_{t+1} | \bar{h}_t, \bar{h}_{t-1}, \dots, \bar{h}_0) &= \\ &= \text{Prob}(C_{t+1} \cap \bar{h}_t, \bar{h}_{t-1}, \dots, \bar{h}_0) / \text{Prob}(\bar{h}_t, \bar{h}_{t-1}, \dots, \bar{h}_0) = \mu(C_{t+1}) = \\ &= (1 - \mu_0) \text{Prob}(D_{t+1} | \bar{h}_t, \bar{h}_{t-1}, \dots, \bar{h}_0) = \\ &= \text{Prob}(D_{t+1} \cap \bar{h}_t, \bar{h}_{t-1}, \dots, \bar{h}_0) / \text{Prob}(\bar{h}_t, \bar{h}_{t-1}, \dots, \bar{h}_0) = \mu(D_{t+1}) = \mu_0. \end{aligned}$$

These transitions describe a simple Markov process with the matrix of transition probabilities presented in Table 4. We let $[1 - \mu_0, \mu_0]$ be the prior vector of probabilities and fix this stochastic matrix for all t .

It is natural to suppose that the longer the pyramid exists, the lower is the sophisticated I-strategist's belief that the firm will not defect in the next period. This is given by the Markov process, stipulated by the usual Bayesian updating: sequential multiplication of the row vector $[1 - \mu_0, \mu_0]$ by the stochastic matrix in Table 4 puts subsequently higher posteriors to D and subsequently lower posteriors to C. Alternatively, the same prior vector may be multiplied by t -step transition matrices calculated via Chapman-Kolmogorov equations, *e.g.*, the transition probabilities matrix after two stages is as shown in Table 5.

Table 4. Matrix of transition probabilities for the beliefs of sophisticated individuals.

| | C | D |
|---|-------------|---------|
| C | $1 - \mu_0$ | μ_0 |
| D | 0 | 1 |

Table 5. Two-step matrix of transition probabilities of beliefs for sophisticated individuals.

| | C | D |
|---|-----------------|----------------------------|
| C | $(1 - \mu_0)^2$ | $\mu_0(1 - \mu_0) + \mu_0$ |
| D | 0 | 1 |

Subsequent replication of the same procedure leads at the end to $\text{Prob}(D) = 1$ which is an absorbing state.

Further, we let individual beliefs vary across individuals. Such heterogeneity can be introduced in the following simple way. In the case of finite N^s , let $\varpi_0 = \min(\mu_0)$ and $w_0 = \max(\mu_0)$ be the highest (below 0.5) and lowest (close to 0) initial subjective beliefs that the pyramid is to defect in the next (*i.e.*, first) period. We assume that the values of μ_0 obey discrete uniform distribution on the $[\varpi_0, w_0]$ segment; the endpoints of this segment will evolve with time as ϖ_t and w_t in an obvious way. If ϖ_0 — the lowest value of μ — is arbitrarily close to 0 (corresponding to those investors who are rational in the Bayesian sense, but do not think the firm is likely to defect at all), then ϖ_t — the minimum of posterior beliefs at t — will remain close to 0, and the behaviour of the holder of such beliefs will resemble that of naive individuals. Confidence of more sceptical subjects will, however, erode faster, and in due time, they will decide to withhold from now and forever. An assumption of the uniform distribution of beliefs in (ϖ_t, w_t) , together with a continuum approximation of beliefs and risk-neutrality, leads to the following sequence of posterior probabilities for the entire class of sophisticated individuals:

$$\begin{aligned} \text{Prob}(\text{Invest at stage } t) &= \text{Prob}[\mu_t < d/(1+d)] = \\ &= [d/(1+d) - \varpi_t]/[w_t - \varpi_t]. \end{aligned} \quad (13)$$

This expression defines the probability that an arbitrarily selected sophisticated individual will prefer I to W. Risk-neutrality implies that the corresponding investment condition is given by

$$(1 - \mu_t)Md - \mu_t M > 0. \quad (14)$$

An individual for whom this condition is not met will choose in favour of W. A threshold value for this inequality depends on parameters d , ϖ_t and w_t , which determine the probability that an arbitrary sophisticated I-strategist will withhold.

Another component of our construction applies to those sophisticated individuals who were playing W because of their ignorance about the pyramid's existence, but became aware of it along with its growth. We formalize this possibility by supposing that in each time period, $N^s/2^t$ individuals are randomly and independently of their beliefs selected without replacement from the population of all sophisticated individuals. These selected players consider an investment prospect which they accept iff (14) happens to be satisfied for them, and those who once checked their preferences, never return to the issue. In other words, we assume that at period 1, $N^s/2$ individuals "learn" about the prospect to

become rich by joining the pyramid. In period 2, this procedure is repeated for half of those who have not been subjected to it in period 1, and so forth, so that at the limit of $t \rightarrow \infty$, should the pyramid survive, everyone will become aware of its existence. In that way, in each period of time at the early stage of the pyramid's growth there will be a uniform "injection" into the current population of I-strategists: the share of those "checkers" who end up investing in every time period is a fraction of $N^s/2^t$ given by (13). However, as $t \rightarrow \infty$, it will happen that $d/(1+d) < \bar{\omega}_t$, i.e., the least sceptical individual will not invest, and the fraction of newcomers will become zero.²²

Finally, let us specify the leakage conditions. First, since sophisticated I-strategists are prudent, they are set to revise their beliefs at the beginning of every time period. For a given individual, as long as (13) is violated, he or she immediately withholds. Conveniently, beliefs of all current I-strategists are uniformly distributed in $(\bar{\omega}_t, d/(1+d))$ for all t . A fraction of such withholders among the current I-strategists for every t is given by

$$s_t = \begin{cases} 1, & \text{if } d/(1+d) < \bar{\omega}_t; \\ \frac{\bar{\omega}_{t+1} - \bar{\omega}_t}{d/(1+d) - \bar{\omega}_t}, & \text{otherwise.} \end{cases} \quad (15)$$

To justify this condition, observe that we need to find the share of current I-strategists n_t^s for whom (14) was satisfied in period t , but is no longer satisfied at $t+1$. From (13) and the uniformity of beliefs in $[\bar{\omega}_t, d/(1+d)]$, n_t^s I-strategists at t were uniformly distributed between $\bar{\omega}_t$ and $d/(1+d)$, while at t , $\bar{\omega}_{t+1} - \bar{\omega}_t$ of them switched to the withholding strategy. This must be true of every consecutive quantity n_t^s until (13) is satisfied for at least some of the sophisticated individuals; however, all of them immediately withhold as long as the minimum of the admissibility region exceeds $d/(1+d)$. Moreover, we want to allow sophisticated individuals to withhold for transactions without strategic purposes, irrespective of their beliefs. We assume this is a fixed fraction c of those who would continue to invest on the grounds of their beliefs.

Suppose that some exogenous number n_1^s of sophisticated individuals happened to invest in period 0. Summarizing the above arguments,

²² We safely rule out the case $w_0 > d/(1+d)$: if sophisticated individuals are "too sceptical" to invest, their behaviour is immaterial for the pyramid's growth.

the discrete-time dynamics of sophisticated I-strategists in periods $t = 1, 2, \dots$ is given by

$$n_{t+1}^s - n_t^s = \frac{N^s}{2^t} \times \max \left\{ 0, \frac{d/(1+d) - \varpi_t}{w_t - \varpi_t} \right\} - n_t^s [s_t + (1-s_t)c_t]. \quad (16)$$

The positive summation above denotes the number of those sophisticated individuals who first consider investing at t and will indeed invest, and the negative one is a leakage of I-strategists for either transaction purposes or reasons of prudence.

C. The dynamics of pyramid's growth

Now we may explicitly combine the two dynamics — (12) for naive and (16) for sophisticated individuals — to create a single dynamics of the population. A relatively nice and compact picture would be obtained by the "vertical sum" of the two dynamics, resulting in

$$\begin{aligned} n_{t+1} &= n_t + \Delta n_t^n + \Delta n_t^s = n_t + 2b q_{I,t}^n [u(e_I, q_t^n) - u(q_t^n, q_t^n)] N^n + \\ &+ \frac{N^s}{2^t} \times \max \left\{ 0, \frac{d/(1+d) - \varpi_t}{w_t - \varpi_t} \right\} - n_t^s [s_t + (1-s_t)c_t], \end{aligned} \quad (17)$$

where, as before, $n_t = n_t^s + n_t^n$ with corresponding indices for quantities related to of N^n and N^s . This picture, however, would not be correct, for in the whole population an arbitrary *naive* W-strategist can meet either a naive or a sophisticated I-strategist, and thus the probability of meeting an I-strategist is approximately q_I , their fractions in the whole population. Accordingly, (11) is to be rewritten as

$$\begin{aligned} \Delta q_I^n &= q_I^n r_I \pi_I^I + (1 - q_I^n) r_W \pi_W^I - q_I^n r_I = \\ &= q_I^n \{1 - (1 - q_I) F[u(e_W, q) - u(e_I, q)] - 1\} + \\ &+ (1 - q_I^n) q_I F[u(e_I, q) - u(e_W, q)] = \\ &= -q_I^n (1 - q_I) \{F[u(e_W, q) - u(e_I, q)] + \\ &+ (1 - q_I^n) q_I F[u(e_I, q) - u(e_W, q)]\}, \end{aligned} \quad (18)$$

of which further simplification is not possible. Recalling our assumption that "beauty is in the eye of the beholder," *i.e.* that utilities of one's own and alternative strategies are estimated from the assessor's personal viewpoint, we still may rely on a linear approximation of $F(\cdot)$ at relatively

low cost: it suffices to assume that the distribution of tastes is on average the same (*i.e.*, uniform with parameters a and b) for either sophisticated or naive individuals. Then (18) is rewritten as

$$\begin{aligned}\Delta q^n &= -q_I^n (1 - q_I) \{F[u(e_W, q) - u(e_I, q)] + \\ &\quad + (1 - q_I^n) q_I F[u(e_I, q) - u(e_W, q)]\} = \\ &= -q_I^n (1 - q_I) \{a + b[u(e_W, q) - u(e_I, q)]\} + \\ &\quad + (1 - q_I^n) q_I \{a + b[u(e_I, q) - u(e_W, q)]\},\end{aligned}\quad (19)$$

which looks of course less neat than (12), resulting in an accordingly complicated expression in place of (17):

$$\begin{aligned}n_{t+1} &= n_t + \Delta n_t^n + \Delta n_t^s = \\ &= n_t - q_I^n (1 - q_I) \{a + b[u(e_W, q) - u(e_I, q)]\} N^n + \\ &\quad + (1 - q_I^n) q_I \{a + b[u(e_I, q) - u(e_W, q)]\} N^n + \\ &\quad + \frac{N^s}{2^t} \times \max \left\{ 0, \frac{d/(1+d) - \varpi_t}{w_t - \varpi_t} \right\} - n_t^s [s_t + (1 - s_t) C_t].\end{aligned}\quad (20)$$

Dynamics (20) is referred as to *basic dynamics* in Section 6. While being rather cumbersome, it still allows us the convenience of working with deterministic dynamics instead of a stochastic one. It also serves as the sufficient statistic for the optimal control problem of the Ponzi firm. Note, finally, that these dynamics of population growth are completely determined by unknown parameters a , b , w_t and ϖ_t which are directly interpretable in economic terms; they correspond to the parameter vector W_t introduced in Section 4.

6. NUMERICAL SIMULATIONS AND ALTERNATIVE SPECIFICATIONS

Empirical tests of the above dynamical models are rather difficult, for no data on the pyramid's growth are readily available. The only publicly available information is the number of "diluted debtholders," n^* , the maximum amount of capital k^* , and also the pyramid's existence time. For instance, according to Russian media, one of the most famous Russian pyramids (Khoher-Invest, Rostov-on-Don) attracted 1500 billion roubles from some 2.5 million private investors; another one (Russian House Selenga, Volgograd) — 2800 billion roubles. As shown in this sec-

tion, these numbers are compatible with the numerical estimations of dynamics (20) and its extensions.

A. Basic dynamics

We used dynamics (20) to simulate pyramid growth with different values of 12 parameters of the basic model — these are d , c , a , b , N^n , N^s , n_0^n , n_0^s , w_0 , ϖ_0 , k_0 , M . For the baseline model, the following parameter values have been used, as stipulated by the experience of Russian pyramids: $N^n = 5000000$, $N^s = 1000000$, ($N = 6$ million people), $q^n = q^s = 0.001$, $d = 0.2$ (thus, $d/(1+d) = 0.167$), $M = 300$, $k_0 = 50000$ (values in thousand roubles, 1994 prices), $w_0 = 0.01$, $\varpi_0 = 0.005$. These numbers seemingly make sense for large-scale pyramids. We tried also different parameters to access comparative static effects as described below.

These values have been affected by the borders of confidence intervals w_0 and ϖ_0 : in particular, lowering of ϖ_0 leads to a slight decrease in n^* and increase in k^* . These are the main parameters that affected the evolution of sophisticated I-strategists as shown in Fig. 2. These dynamics have a single peak and decline to zero when the pyramid's lifetime tends to infinity; however, since the game is terminating in finite time,

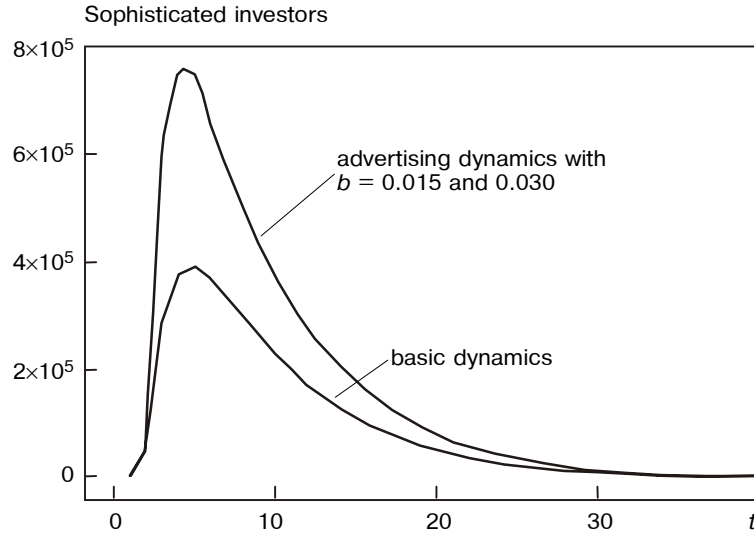


Fig. 2. Dynamics of sophisticated investors (in the absence of defection).

they will exhibit a break as shown in Fig. 3. These dynamics are not very sensitive to parameter values: the lower (solid) line shows the pattern for standard dynamics (20); the upper (dashed) line corresponds to an alternative specification described below.

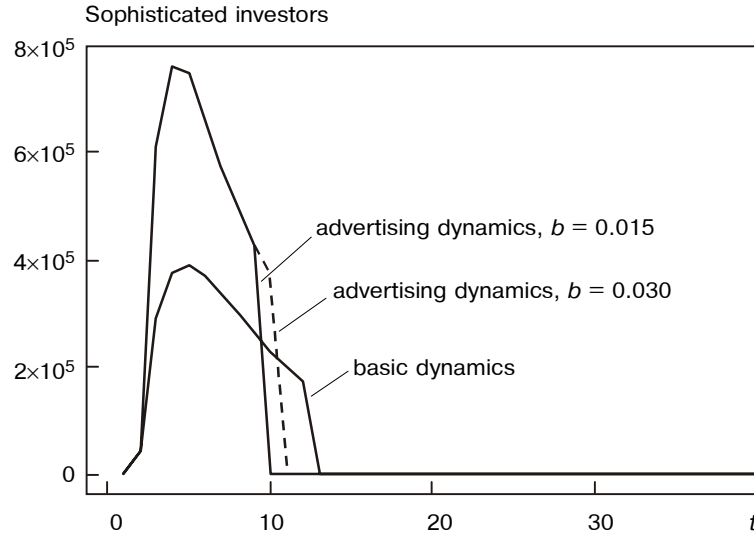


Fig. 3. Dynamics of sophisticated investors (with defection).

Not surprisingly, higher value of c leads to the lowering of n^* and k^* , and also serves to extend the lifetime of the pyramid — in our estimations we used $c = 0.1$. The dynamics of the pyramid's growth was also not sensitive to the initial value of capital, implying that the modern Ponzis firm might start with virtually no fixed costs, and that the confidence of the public was the principal source of its success.

The main determinants of the dependence of these dynamics of time are unobservable parameters a and b , which have had to be fitted. Since all utilities are understood in the sense of Neumann–Morgenstern, *i.e.*, defined up to affine transformations, the value of a is deprived of meaning — we set it equal to zero. By contrast, b is meaningful: the higher it is, the higher are the chances that the individual, observing the more profitable strategy of another player will find it worthwhile to switch. One may also say that this parameter indicates the elasticity of an individual's reaction across the population to the observed performance of "the other guy" in an auxiliary game (Table 3). Higher b implies this is very likely, for our baseline model

$b \in (0, 0.033)$ results in proper probabilistic dynamics of naive individuals as shown in Fig. 4. If the value of b is below 0.01, the population of naive I-strategists initially grows, but then declines and is wiped out. At middle levels of b at about 0.010 to 0.025, the dynamics of the share of I-strategists reaches a steady state that depends on b ; initially q_I comes to exceed this steady state, but gradually q_I returns back to it from above. At higher levels of b , the dynamics of *naive* I-strategists become more volatile, and explode at values above 0.033. A vertical sum of the dynamics of naive I-strategists for four alternative values of b (0.005, 0.015, 0.025 and 0.030) and for that of sophisticated individuals is shown in Fig. 4.²³ These patterns are typical, and clearly higher b leads to greater and faster growth in the pyramid and higher k^* as shown in Fig. 5. It follows from the last two pictures that when b is higher, the stopping time T is reached earlier: it varies normally between 4 and 8,

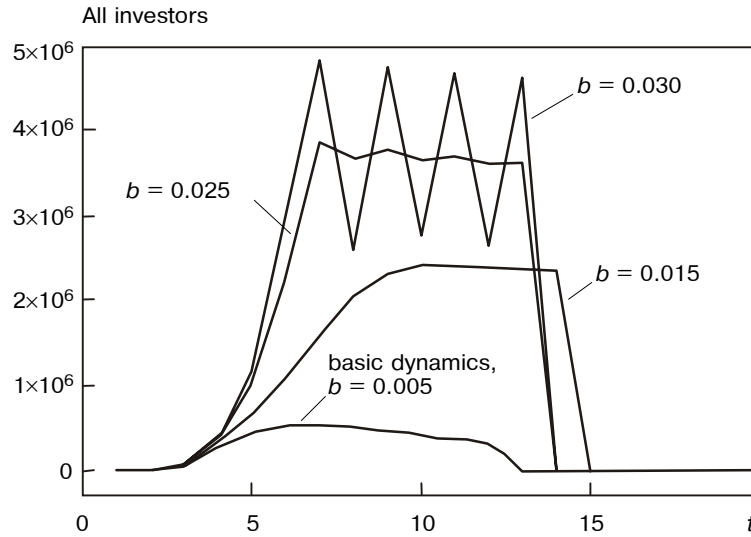


Fig. 4. Basic dynamics of I-strategists (naive and sophisticated).

²³ Symptomatically, n_t with the lowest $b = 0.005$ follows the dynamics of sophisticated individuals, which has a tendency to hold for *naive* individuals alone. This is because naive individuals then switch from W to I with high caution, and if the subpopulation of sophisticated individuals is not negligible, the number of such switches will depend on what these latter are doing. In other words, with low b , the mechanism of auto-reproduction of *naive* individuals on their own is not "turned on."

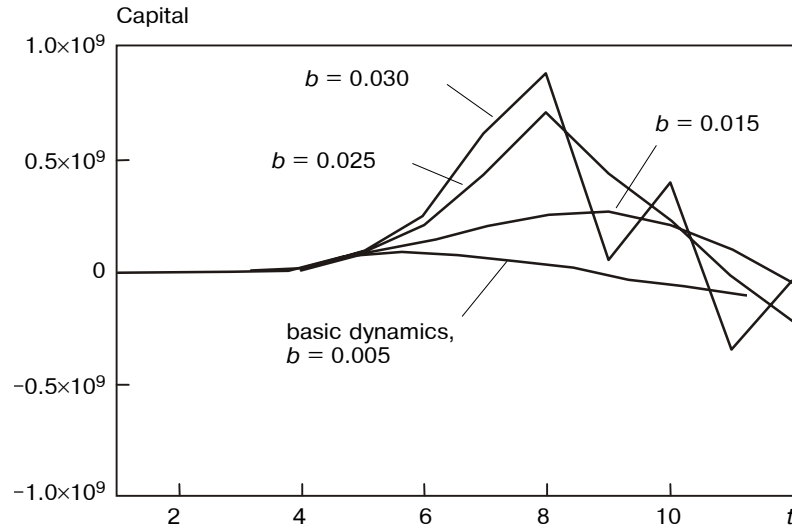


Fig. 5. Basic dynamics of capital.

although under some combinations of parameters ($c = 0.5$, $d = 0.1$, $b = 0.005$) it may extend to 12. Values of n^* and k^* are shown in Table 6 for the basic and other models.

Another interesting question is whether the composition of the population (*i.e.* shares of naive and sophisticated individuals in N) matters. When most subjects are sophisticated, the dynamics of the population closely replicates that of their subpopulation (Fig. 3). Since naive individuals mostly meet sophisticated ones, their subpopulation also follows the same dynamics: symptomatically, the share of naive I-strategists reaches its maximum at T . By contrast, a higher proportion of naive individuals (Table 6: naive) returns the dynamics under which the share of naive individuals grows faster and sharper when b is larger. In other words, Ponzi manage to "extract" more money from naive individuals before the collapse. Notice that the share of naive I-strategists for mid-valued b 's tends to some steady state. This implies that the population of investors in a long-lived investment opportunity (like the Russian financial assets, GKO) will tend to stabilize.

The main factor affecting the value of k^* is M . The average size of a deposit allows one to attract much more money, and also tends (via replicator dynamics) to extend the pyramid's lifetime (only two values are shown in Table 6; higher values of b have to be ruled out because of a lower instability threshold). Some data available suggest that the average

Table 6. Characteristics of the pyramid at the optimal stopping time for different parameter values.

| Parameters | n^* , thousand people | k^* , bln Rubles (1994 r.) | T |
|----------------------------|-------------------------|------------------------------|-----|
| Basic, $b = 0.005$ | 533 | 87 | 6 |
| Basic, $b = 0.015$ | 2419 | 266 | 8 |
| Basic, $b = 0.025$ | 3668 | 706 | 8 |
| Basic, $b = 0.030$ | 4867 | 887 | 9 |
| Naive, $b = 0.005$ | 540 | 88 | 6 |
| Naive, $b = 0.015$ | 2840 | 307 | 9 |
| Naive, $b = 0.025$ | 4446 | 819 | 8 |
| Naive, $b = 0.030$ | 5753 | 1078 | 8 |
| Deposit, $b = 0.005$ | 1550 | 359 | 8 |
| Deposit, $b = 0.015$ | 5530 | 1980 | 7 |
| Interest low, $b = 0.005$ | 583 | 136 | 6 |
| Interest low, $b = 0.015$ | 897 | 188 | 7 |
| Interest low, $b = 0.025$ | 1900 | 353 | 8 |
| Interest low, $b = 0.030$ | 2511 | 485 | 8 |
| Interest high, $b = 0.005$ | 668 | 55 | 7 |
| Interest high, $b = 0.015$ | 3699 | 367 | 8 |
| Interest high, $b = 0.025$ | 5132 | 718 | 8 |
| Advertising, $b=0.015$ | 3192 | 538 | 5 |
| Advertising, $b=0.030$ | 4379 | 827 | 5 |

Specifications of the dynamics are as follows (see text for general description):

naive: basic + $N^n = 5900000$, $N^s = 100000$, $N = 6000000$;

deposit: basic + $M = 700$;

interest low: basic + $d = 0.1$;

interest high: basic + $d = 0.3$;

advertising: $\xi = 0.6$.

amount of deposits in the Russian 1994 pyramids could be closer to $M = 700$ than to $M = 300$; however, in this case the firm's task of finding T is especially difficult even under deterministic dynamics of the pyramids' growth. It follows from our analysis that the crucial factor of T 's

estimation is the value of b , which itself is an unobservable psychological characteristic of the population. Our model, however, reveals some of these psychological characteristics: thus, as shown by Figs 4 and 5, the full story of the pyramid is not necessarily over at the maximized value of capital. The investors' population is predicted to decline only gradually after the firm is unable to attract new deposits. Even after the firm is ruined ($k_t < 0$), the proportion of its naive investors does not immediately drop to zero. This property of the replicator dynamics captures the fact that it takes time for the population of naive individuals to accept the end of their dream related to the pyramid.

On the firm's side, the natural aim is to try to predict and control parameter b . One natural mechanism for doing so is interest rate d whose effects are also shown in Table 6. Lower values of d smooth the pyramid's growth and increase the pyramid's lifetime with higher b , but drastically decrease its size. Higher d induces faster growth in the gradual outflow of I-strategists lead by cautious sophisticated investors until the pyramid collapses; however, even in most profitable case of $b = 0.025$, the value of k^* is lower than in the basic case because of higher outlays. Since very high levels of d will lead to explosive dynamics and may not look trustworthy, average values of d (say, 10 to 20% over the market rate) look optimal, and were indeed by far the most common in reality. In both cases, smoothing of the dynamics occurs at a rather high cost; a less expensive way to reach the same aim will be considered in the next subsection.

B. Advertising campaign

The first obvious extension of the baseline model (20) consists of the use of alternative stimuli, such as an advertising campaign. It seems natural to suppose that the efficiency of this campaign (in terms of attracted deposits) declines with time, for N is finite. At the same time, one may want the intensity of the campaign to depend on the current firm's resources. We find it convenient to evaluate the efficiency of the campaign by the share of those withholders at stage $t - 1$ who converted to investors at t . Of the many possible evolutions of this share, consider the following dynamics:

$$\Delta q_I^* = \frac{\exp[\kappa_1 t(k_t - k_{t-1})/k_t] \kappa_2 (k_t - k_{t-1})/k_t}{\Gamma(t)}, \quad (21)$$

where κ_1 and κ_2 are parameters and $\Gamma(t)$. Euler's gamma function is used to normalize values of q_I to the unit interval. The intuition behind this specification is that the Ponzi managers agree to spend the fixed

share κ_1 of the last increment of capital. In such a case, the higher was this increment in the last period, the higher will be the amount allocated to advertising. Division by k_t in (21) is again a matter of normalization, corresponding also to the fact that the higher k_t is, the lower are the firm's incentive to advertise further. The cost of this campaign is then $\kappa_1(k_t - k_{t-1})$, but its efficiency will vary with parameter κ_2 : the higher it is, the more efficient is the campaign. Altogether the dynamics (21) are similar to a gamma distribution: it exhibits a sharp increase at first, and an exponential decline afterwards. Lower (closer to 0) values of κ_2 dampen these dynamics; in our estimations we have assumed that the campaign was efficient, and set $\kappa_2 = 1$. With $\kappa_1 = 0.2$ and $a = 0$, (20) is rewritten as

$$\begin{aligned}
 n_{t+1} - n_t &= \Delta n_t^n + \Delta n_t^s = \\
 &= -q_t^n (1 - q_t) \{b[u(e_W, q) - u(e_I, q)]\} N^n + \\
 &+ (1 - q_t^n) \{\xi q_t b[u(e_I, q) - u(e_W, q)] + (1 - \xi) \Delta q_t^* \} N^n + \\
 &+ \Delta q_t^* (N^s - n_t^s) \max \left\{ 0, \frac{d/(1+d) - \bar{w}_t}{w_t - \bar{w}_t} \right\} - \\
 &- n_t^s [s_t + (1 - s_t)c]. \tag{22}
 \end{aligned}$$

These *advertising dynamics* (in contrast to the basic dynamics, (20)) assume that the inflow of the naive I-strategists is affected by two components: the first of which ($\xi \in [0, 1]$) is due to the imitation of successful behaviour as above; and the second ($1 - \xi$) is the result of advertising as specified by (21). The same dynamics are substituted for the fraction of sophisticated individuals whose dynamics are shown in dotted lines in Fig. 3 for two values of $b = 0.015$ and 0.030 . Symptomatically, both lines are the same at the beginning, and only slightly decrease at the top of the pyramid, corresponding to the fact that higher b incentives to advertise decline, but only at higher values of k_t .

The dynamic paths of (21) compared to (20) for these two values of b are shown in Figs 2, 6, 7 and 8; see Table 6 for comparison of values of n^* , k^* and T . At least three important tendencies are worth mentioning. First, the peak of investors and capital is reached faster, by period 5, which is synchronised for different values of b . Second, fluctuations of capital for higher values of b are dampened, though not eliminated. These two features imply that efficient advertising indeed helps to control the dynamics of population strategies. Third, k^* — the optimum value of capital is higher than before for $b = 0.015$, but lower than before for $b = 0.030$. This implies that advertising is efficient for the Ponzi firm if

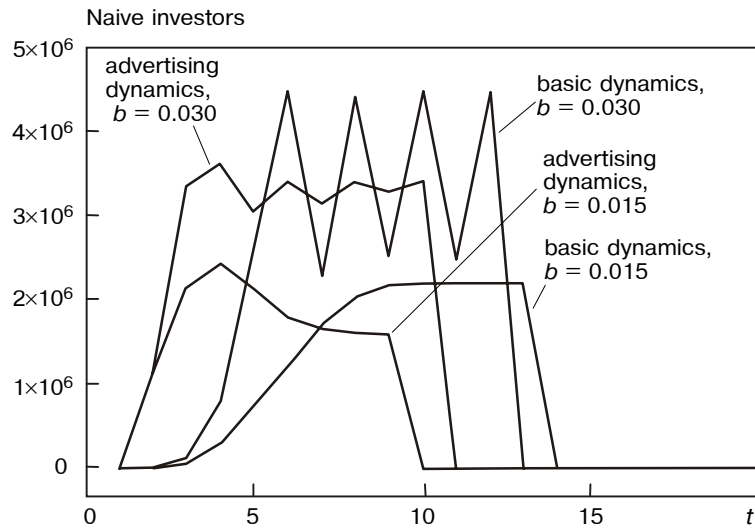


Fig. 6. Basic vs. advertising dynamics of naive investors.

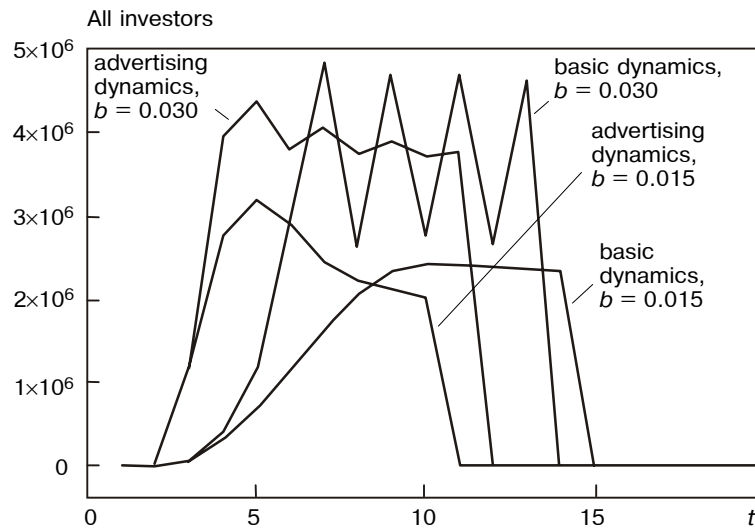


Fig. 7. Basic vs. advertising dynamics of I-strategists (naive and sophisticated).

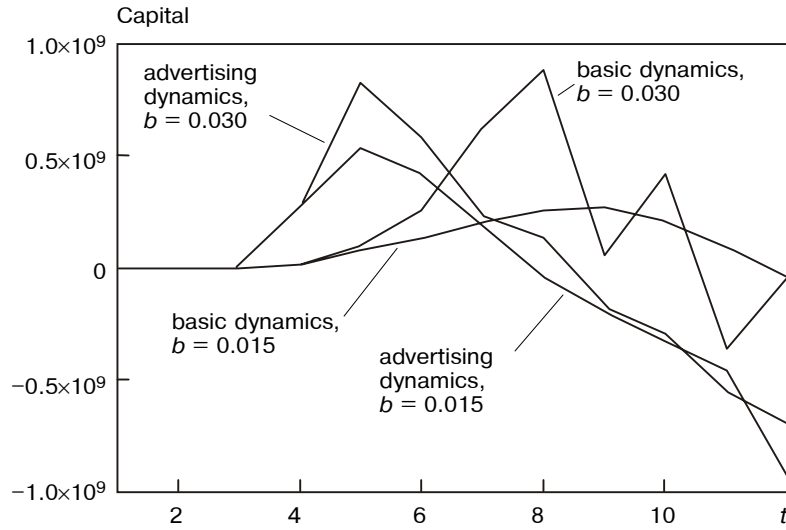


Fig. 8. Basic vs. advertising dynamics of capital.

the elasticity of individuals' reactions to others' performance is quite low (and advertising improves it) rather than when it is high (and advertising just constitutes an extra cost).

C. Effects of nonlinear utilities

Another possible generalization of dynamics (20) arises if individuals would maximize their expected utilities $\sum u(x)p_x$, or generalized expected utilities $\sum u(x)g(p_x)$, instead of expected value. In this broader class of functionals, the rank-dependent utility models (Quiggin 1982; 1993; Yaari 1987; Wakker 1994) are the most natural candidates; however, the effects of such modification are straightforward. For, in either case, the dynamics of the pyramid's growth will be dampened if individuals are *risk-averse*, and have a concave utility function (in the case of expected utility) or concave probability weighting function g (in rank-dependent expected utility, in which case g 's are capacities). We omit numerical simulations of these effects, for they are pretty similar to those considered above, and we find it very difficult to suggest any specific nonlinear transformation of the outcomes or probabilities that could be more meaningful than the others.

D. A simpler derivation of replicator dynamics

It has been noticed before that a strategically minded sophisticated I-strategist faces the stochastic control problem of a prohibitively complicated structure. This suggests that (nearly) all individuals may have to use some simple clues upon which reasonable individuals could condition their strategies. In view of this possibility, we now relax the naive-sophisticated taxonomy in favour of an alternative rationality specification. Assume that the behaviour of all members of a large population is strategically indistinguishable, but on average all individuals approaching the Ponzi game are 1) prudent, *i.e.* don't want to risk too much, and 2) justify their strategies upon comparing it to other's behaviour, *e.g.* through the mechanism of symmetric auxiliary game from Table 3. In this section we construct an example of such behavioural motivation *for the particular case* of the Ponzi game, which directly leads to a variant of the replicator dynamics (6).

Consider again an auxiliary game from Table 3, and let the population of N individuals be partitioned to I- and W-strategists in proportions q_I and $1 - q_I$, respectively, at any moment in time. In an auxiliary game, a randomly drawn I-strategist will meet another I-strategist with probability $(N^n - q_I N^n - 1)/(N^n - 1)$, and meet a W-strategist with probability $q_I N^n/(N^n - 1)$. If N is sufficiently large, these quantities can be approximated by the current proportions of strategies I and W, $1 - q_I$ and q_I , respectively. Further, we explicitly suppose that random matches of players now happen in continuous time. Taking a sufficiently small time period, the probability of having more than one individual learning within the same time period may be considered as negligible (Samuelson, 1997), a fact that allows us to concentrate on the expected outcome of a single match.

An arbitrarily drawn individual can be either a current W-strategist (with probability $1 - q_I$), or a current I-strategist (with probability q_I). In the former case, he will have no chance to change the strategy to I unless he meets an I-strategist, which probability is given by q_I — the current fraction of I-strategists. In the case of this event, let the probability that a W-strategist will desire to switch to strategy I be given by function φ_I : $u(e_I, q) - u(e_W, q) \rightarrow [0, 1]$. The domain of φ_I is the *difference* between $u(e_I, q)$ and $u(e_W, q)$, so that he will switch from W to I if this difference is positive (utilities may be allowed to be subject-specific, as in Section 5a, but their range must be given). The range of φ_I has to be $[0, 1]$; furthermore, we assume it takes the simple form: $\varphi_I = \beta u(e_I, q) - \alpha u(e_W, q)$, with properly selected coefficients α and β that bound φ_I to the unit interval for every individual. These coefficients support the following inter-

pretation: β captures the extent to which individuals react to the firm's signals (e.g. high interest rate or advertising campaign), while α indicates the intensity of individuals' regular market operations, which is proportional to the individual's demand for money. Thus, the higher β is and the lower α is, the more likely is the current withholder (one of $N(1 - q)$) to invest in the firm once *thought of this opportunity* which thought is conditional upon meeting a current I-strategist.

Thus the probability that an arbitrary W-strategist will switch and invest, or the expected injection to strategy I, is given by

$$(1 - q_I)q_I [\beta u(e_I, q) - \alpha u(e_W, q)]. \quad (23)$$

The expected leakage from this strategy applies to current I-strategists only. We assume they switch to withhold unconditionally upon meeting a withholder for two basic reasons. First, they may simply need money for regular transaction purposes, which is more likely the higher is coefficient α . This necessity is counterbalanced by higher incentives to keep money in the firm, as captured by β . Withdrawal of a current deposit can then be captured by $\alpha u(e_I, q) - \beta u(e_I, q)$, which also should have its range in $[0, 1]$. This would obviously require $\alpha > \beta$; moreover, it must be coordinated with the use of these coefficients in (21).

A final component allows us to take into account the individual's precaution when investing into the financial pyramid. One way to capture this is to stipulate that when the proportion of the population currently playing I is high, the subject is more likely to be afraid of its collapse, and thus will tend to withhold. This caution, then, may be measured by the fraction of current I-strategists q_I . Altogether, the probability that a single individual plays I and will switch to W is

$$q_I q_I [\alpha u(e_I, q) - \beta u(e_I, q)], \quad (24)$$

and the expected increment of I-strategists is then approximated by the difference between (21) and (22), namely,

$$(1 - q_I)q_I [\beta u(e_I, q) - \alpha u(e_W, q)] - q_I q_I [\alpha u(e_I, q) - \beta u(e_I, q)]. \quad (25)$$

Rearranging this,

$$\begin{aligned} (1 - q_I)q_I \beta u(e_I, q) - (1 - q_I)q_I \alpha u(e_W, q) - q_I q_I \alpha u(e_I, q) + \\ + q_I q_I \beta u(e_I, q) = q_I [(1 - q_I) \beta u(e_I, q) + q_I \beta u(e_I, q) - \\ - (1 - q_I) \alpha u(e_W, q) - q_I \alpha u(e_I, q)] = q_I [\beta u(e_I, q) - \alpha u(q, q)], \end{aligned} \quad (26)$$

We obtain a replicator dynamics equation with parameters β and α . The specification just described requires the population to be homogeneous (or deals with the populations' averages), but it also has the advantage

of allowing a separated analysis of four factors' effects: probability of defection; utilities of various outcomes; liquidity preference α ; and elasticity of *deposits on incentives* β . With the same parameter values as above, we were able to obtain appropriate coefficients α and β with ranges $(1/6, 1/3)$ and $(0, 1/6)$, respectively, and an additional condition that the difference between particular values assumed by α and β is less than $1/6$. With these parameters, the function φ_i meets the requirements made above, and we obtain dynamics (24) which, with zero probability of the firm's defection, closely repeats those with mid-valued parameter b for model (20), but with more distant satiation (see Figs 9 and 10, where $\alpha = 0.25$, $\beta = 0.15$, inflow component in (23) amounts to 0.9 and outflow in (24) amounts to 0.6. Collapse of the pyramid again occurs, but later than in the basic case.

7. CONCLUSION

The issues discussed in this paper are pertinent to the puzzling and important phenomenon of transitional economies — that of financial pyramids. We found that, despite some features common to standard games, such as reputation and rationalizability, they are essentially disequilibrium phenomena. We have shown that this game has no (nontrivial) equilib-

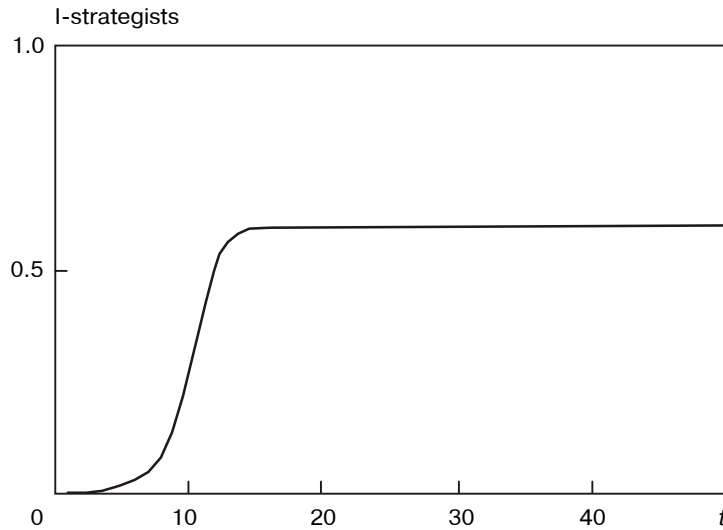


Fig. 9. Alternative I-strategists dynamics.

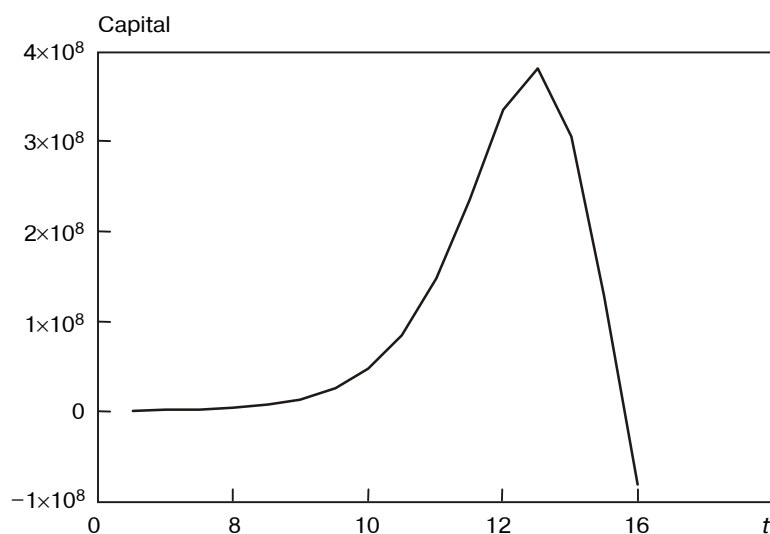


Fig. 10. Alternative capital dynamics.

rium, and that any cautious strategy may be successful only if by chance and at the expense of other fellow individuals. Symptomatically, this strategy also cannot be justified on the grounds of the rule of long-run success in the sense that were the individual invest in a sufficient number of pyramids whilst these were growing, he or she would benefit on average. A fallacy of this type of argument has been unveiled by Samuelson (1963), who has shown that no expected-utility maximizing individual should accept a sequence of equivalent fair gambles if she rejects a single game of the same kind. Accordingly, if no optimal rules based on the principle of expected-utility (or expected-value) maximization exist for a single pyramid, no such rules can be designed for a diversified strategy either.²⁴ Finally, in reality there is no guarantee that the firm would be rational enough to follow its optimal policy; a single "tremble" would destroy any equilibrium calculations of the smartest of the individuals. It follows from this paper that the real worth of investing in the Ponzi firm is rather like a casino bet using a random device with chances that are only known to be unfair, the only difference being that the outcome is revealed faster in this latter case.

The only meaningful advice to the individuals who want to be called 'rational' and think of playing Ponzi games is thus not to mess with them;

²⁴ This result, however, is not valid for a broad class of generalized EUT functionals (Segal and Spivak, 1988).

nevertheless, in reality many did. This raises the following, natural questions: 1) which economic circumstances have created incentives for them to do so, paving the way for the pyramids' growth; and 2) how did the market for savings evolve after having experienced Ponzi games, and in particular, what did this experience teach them. We find it likely that the instruments proposed and used in this paper shall be helpful in addressing these questions within the framework of multi-stage signalling games, which leads to the construction of appropriate population dynamics. These tasks are left for the future work.

APPENDIX. PROOFS

Proof of Propositions 1 and 1a

We consider the evolution of a subpopulation of I-strategists as a homogeneous Markov chain with finite state space S and a given stochastic matrix of $p_{S'S, t+1}$. Given the finiteness of the set of all possible histories, $\{k_t\}$ is a sequence of well-defined random variables — values of capital at each time period. Values of k_t for different histories may coincide, reflecting the fact that it doesn't matter for the firm which particular individual has played W or I at every t ; however, for every history there corresponds a unique sequence $\{k_t\}$. It suffices, therefore, to limit our attention to payoff-relevant histories, *i.e.*, to those partitions of H_t that are equivalent from the viewpoint of the firm's payoffs (Fudenberg and Tirole, 1991, ch.13). The firm's task is to find the Markov moment t^* ²⁵ for the process $\{k_t\}$ (The Markov moment is called stopping time T if t^* is reached in finite time with probability 1.).

Consider first the case of a unit discount rate and a risk-neutral capital maximizing firm whose task it is to find stopping time T (*i.e.*, set $t^* = T$) for the value of t that solves

$$\max E_{n(t)} \left\{ \left[\sum_{t=1}^{T-1} k_{t-1} + \Delta n_t M - n_t M d \right] - [\Delta n_t M - n_t M d] \right\}; \quad (A1)$$

Here, expectation is to be taken at the beginning of every period and is conditional upon the firm's sufficient statistic $h_{N+1, t}$, which may lead to a large variety of particular solutions. At stopping time T , the expected increment of capital $\Delta n_T M$ should be lower than the interest payments due, $n_T M d$, so that the whole expression in the second square bracket should be negative. This expression will represent the firm's foregone cost when defecting at T . Using the Bellman optimality principle, at any prior-to-last stage it is optimal to maximize the sum of expected value of capital at the current stage and expected value of capital at all subsequent stages, provided the optimal policy is used at all stages. Let V_t denote the maximum value of capital at every period: thus, V_T is the maximum value of

²⁵ The Markov moment is a random variable t^* defined with respect to a nondecreasing sequence of algebras whenever the set $\{t^* \leq t\}$ is measurable for every t .

capital (2) at the last stage T , V_{T-1} is its maximum value at $T-1$:

$$V_{T-1}(k_{T-1}) = \max E_{n(T-1)}\{[k_{T-2} + \Delta n_{T-1}M - n_{T-1}Md]\}, \quad (A2)$$

and so forth. By the responsiveness condition and the definition of stopping time, $T-1$ is the last moment when it is optimal to play C; in the next period T , the firm should stop the Ponzi game by defecting. Proceeding backward and making use of additive separability,

$$\begin{aligned} V_{T-2}(k_{T-2}) &= \max E_{n(T-2)}\{k_{T-3} + \Delta n_{T-2}M - n_{T-2}Md\} + \\ &+ \max E_{n(T-2)}\{V_{T-1}[k_{T-1}]\} = \\ &= \max E_{n(T-2)}\{k_{T-3} + \Delta n_{T-2}M - n_{T-2}Md\} + \\ &+ V_{T-1}[k_{T-2} + \Delta n_{T-1}M - n_{T-1}Md], \end{aligned} \quad (A3)$$

since $k_{T-1} = k_{T-2} + \Delta n_{T-1}M - n_{T-1}Md$, and the optimal value V_{T-1} should be added to any value of k_{T-1} at $T-1$. By responsiveness, the optimal decision at stage $T-2$ is to cooperate as well. Continuing recursively backward, we see that the same strategy will ensure that V_t is an optimal value of capital for period t and all subsequent periods $t+1, \dots, T$ due to the enforcement of the optimality principle. It is easy to see that the specification of Proposition 1a does nothing but removes some ambiguity concerning the stopping time: if the dynamics of investors are monotonic, the firm simply monitors their inflow, and defects as long as it notices the inflow of new investment weakening.

Returning now to a more general case, we relax the assumptions of risk neutrality and of no discounting. This will lead to the specification

$$\max E_{n(t)}u_{N+1}\left[\sum_{t=1}^{T-1}\delta^{t-1}k_{t-1} + \Delta n_t M - n_t Md\right] - E_n u_{N+1}[\Delta n_T M - n_T Md], \quad (A4)$$

where $u_{N+1}(\cdot)$ is the firm's utility function for money, and δ is the discount factor applied to the capital of the forthcoming periods. This functional form, in particular, emphasises the fact that most of the modern Ponzis wanted to accumulate money not for any productive activities within the economy, but just for their own consumption. Q.E.D.

Proof of Proposition 2

For reader's convenience, recall that an equilibrium profile q is called.

- (Trembling-hand) **perfect** if there exists $\{q^i\}$ — a sequence of completely mixed strategies converging to q in the strategies' space, s.t. $u_F(q_F, q_{-F}^i) \geq u_F(e_F, q_{-F}^i)$, $\forall F, \forall e_F \in Q_F$ (spanning over the set of pure strategies, e_F is sufficient). An alternative formulation of the perfection

of q (due to Myerson) requires profile q to be any limit (with $\varepsilon \rightarrow 0$) of " ε -perfect" equilibrium profiles of completely mixed strategies q^ε s.t. $u_F(e_F, q_{-\ell}^i) < u_F(e'_\ell, q_{-\ell}^i) \Rightarrow q_\ell^\varepsilon(e_F) < \varepsilon$.

- **Proper** if it is a limit of any " ε -proper" equilibrium profile q^ε of completely mixed strategies s.t. $u_F(e_F, q_{-\ell}^\varepsilon) < u_F(e'_\ell, q_{-\ell}^i) \Rightarrow q_\ell^\varepsilon(e_F) < \varepsilon q_\ell^\varepsilon(e_F)$.
- **Essential** if $\forall \varepsilon > 0 \exists \eta > 0$ s.t. for all games $u'(q)$ perturbed about $u(q)$ with payoffs no more distant than η , there exists an equilibrium profile q' which is no more distant from q than ε (all distances being measured with respect to the standard Euclidean metric).
- **Sequential** (based on an extensive form) if it is *sequentially rational*, i.e., (a subject and verb is missing here or "expected" utility is maximized...) expected utility-maximising for all players, given their posterior beliefs and reached information sets; and *consistent*, i.e., there is a limit of some sequence of completely mixed strategies and posterior beliefs updated whenever possible by the Bayes rule.
- **Stable** (set-valued notion) if this equilibrium set is *closed* and *minimal* with respect to the following property: $\forall \eta > 0 \exists \varepsilon > 0$ s.t. $\forall \varepsilon' < \varepsilon$; any profile of completely mixed strategies $\varepsilon'(e_F)$ (denoting the maximum allowable trembles for all players and strategies), has an equilibrium within η of that set (in the set of strategies).

For a comprehensive discussion of these and other refinements, see van Damme (1991). (Note that application of these refinements to our case require a continuum approximation of payoffs of the game from Table 1.)

Proof: In a stage-truncated game from Table 2, the firm's pure strategies C and D can be dominant for appropriate types, and the dominance is not upset by a single individual's tremble to W or I (this assumption is not crucial for the following proofs). To show that the equilibrium component with the firm's strategies $\theta_t^+ C$ will contain trembling-hand perfect equilibria, take any of these and consider a decreasing sequence of trembles in completely mixed strategies with cumulative probabilities of playing defective strategies below $\varepsilon = 1 - [1/(1+d)] = d/(1+d)$. When probabilities of such trembles are below ε , I is the best response for the individual. Conversely, any sequence of completely mixed strategies with a cumulative probability of tremble to C below $\varepsilon' = 1/(1+d)$ will result in trembling-hand perfection of equilibria in $(\theta_t^- D, W)$.²⁶ Properness too

²⁶ Note, however, that the tremble's "allowance" is much higher in the (C, I) than in the (D, W) case. Both cases also directly follow from the fact that none of these equilibrium strategies are weakly dominated under their corresponding conditions.

follows from a similar argument, but requires smaller probabilities of disequilibrium trembles, together with finite disutility of $-M$ and finite utility of dM for equilibria in I and W , respectively. Limiting the above argument to a sufficiently small closed subset of each equilibrium component, together with sufficiently small trembles, would return the stable set. For any ε -perturbation of mixed strategies there will be a bounded perturbation of payoffs that will preserve fixed points of best-reply correspondences within given ε , which is essential. Finally, the sequentiality is directly implied by perfection for any extensive form corresponding to the normal form from Table 2. Furthermore, applying the same reasoning of a θ_t^+ C component to a collection of several truncated games based on γ_t , perfectness and sequentiality will hold for these profiles over a number of periods (we do not state this observation as a separate proposition). Q.E.D.

Proof of proposition 4

The required universal belief space under construction is conceived as a sequence of spaces $\Omega_1, \Omega_2, \dots$ for the stage-games at periods $t = 0, 1, \dots$ along the lines of Brandenburger and Dekel (1993), whose approach is somewhat simpler than that of Mertens and Zamir, and makes explicit use of the Kolmogorov extension theorem. We construct a complete type space for the initial stage-game γ_0 of the game Γ , followed by the gradual "removal" of those events that did not occur by each consecutive time period. Assume that the firm's capital takes values on some bounded subset of the positive half of the real line, denoted by K_0 , and that the original strategy of the population of individuals is characterized by parameter vector W_0 . The set of all values of $K_0 \times W_0$ is assumed to be a compact²⁷ Polish (non-empty, complete, separable metric) space. Since the strategies' space is fixed and known, and since payoffs to every individual player are the same across stages, denoted by $I = \{Md, M, 0\}$, we observe that the space of possible "states" of the game γ_0 is

$$K_0 \times W_0 \times \prod_{\ell=1}^N I_{\ell}$$

— a compact Polish space.

²⁷ The compactness assumption is hardly too strong in this context, for the set of the firm's possible strategies is finite ($|Q| = 2$), the population is finite, and possible values of the firm's capital are bounded from above from all individuals' viewpoints. Compactifying these spaces if necessary, their $N + 1$ product is compact by Tikhonov's theorem.

Introducing now the time dimension, notice that since the population is finite, and individuals' payoffs are the same throughout the game Γ , for all $t < \infty$ the product

$$\times_{t=0}^{\infty} (K_t \times W_t \times \prod_{\ell=1}^N I_{\ell})$$

will be the space of "physical uncertainty", also compact Polish by construction. This last space of all possible histories of play in the game Γ is still too large; to ensure its inner consistency, attention should be limited to its subspace satisfying the following *structural condition*: for every period t ,

$$K_{t+1} \times W_{t+1} \times \prod_{\ell=1}^N I_{\ell} = \rho_{s',t}[(K_t \times W_t \times \prod_{\ell=1}^N I_{\ell}) \times q_t]$$

for all functions $\rho(\cdot)$ corresponding to every point in

$$K_t \times W_t \times \prod_{\ell=1}^N I_{\ell}$$

and every q_t . (Note that in view of the finiteness of possible profiles, the transition functions ρ are always measurable and well-defined.) The sequence of these subsets indexed with t , $\{B_t^0\}$ is a subspace of the "physical uncertainty" space we shall be dealing with.

For any stage-game t , $P(B_t^0)$ is the set of all probability measures on B_t^0 endowed with a weak topology, which is just sufficient to guarantee that any sequence of probability measures converges to some probability measure if and only if $\int f dp_n \rightarrow \int f dp$ for every bounded continuous function f defined on B_t^0 . The set $P(B_t^0)$ is also a compact and Polish space; it encompasses all beliefs held at stage t by all N individuals and the firm. The Cartesian product of these belief spaces for all individuals,

$$B_t^1 \equiv B_t^0 \times \prod_{\ell=1}^N P_{\ell}(B_t^0),$$

is called the 1-level space,²⁸ and its points are 1-level beliefs, defined as

²⁸ In our specific case, both the firm and all individuals are uncertain about point W_0 , and all individuals, in addition, about point K_0 . However, at level B_t^1 , both of these uncertainties become valid to every player. For notational simplicity we use the general format at the outset.

$\theta_{it}^1 \in \Theta_{it}^1 \equiv P_\ell(B_t^1)$. Proceeding inductively for the spaces of levels $n = 2, 3, \dots$,

$$B_t^n = B_t^{n-1} \times \prod_{\ell=1}^N P_\ell(B_t^{n-1}), \dots,$$

we obtain every n -level belief space as the product of individual beliefs over B_t^{n-1} , denoted as

$$P\left(B_t^{n-1} \times \prod_{\ell=1}^{N+1} P_\ell(B_t^{n-1})\right).$$

A type $\theta_{it} \in \Theta_{it} \equiv P_F(B_t^\infty)$ of every player is just this infinite hierarchy of beliefs — a point in the

$$B_t^0 \times \left(\bigtimes_{n=1}^{\infty} \prod_{\ell=1}^{N+1} (\Theta_{it}^n) \right)$$

space.

Call the type $\theta_{it} = \{\theta_{it}^1, \theta_{it}^2, \theta_{it}^3, \dots\} \in \Theta_{it}$ *coherent* if the marginal distribution of θ_{it}^n on the space B_t^{n-2} coincides with a marginal distribution of θ_{it}^{n-1} on Θ_{it}^{n-2} for all levels $n > 0$ and $n - 1$. This condition requires that individuals' beliefs about the true physical state and/or types of his opponents of previous levels do not change at any higher level. Since not all beliefs are necessarily coherent, this restriction bites, limiting attention to a (clearly compact) subset of

$$B_t^0 \times \left(\bigtimes_{n=1}^{\infty} \prod_{\ell=1}^{N+1} (\Theta_{it}^n) \right) \equiv \bar{\Theta}_{it};$$

note that by construction it incorporates all possible future paths of the game.

By the standard result from the probability theory, the coherence condition as formulated above is necessary and sufficient for the Kolmogorov extension theorem for the $(\bar{\mathcal{O}}^\infty, \bar{\mathcal{O}}_0(\bar{\mathcal{O}}^\infty))$ space (and for the stochastic process²⁹). By the coherence property, there exists a surjective map

²⁹ For this representation, it actually suffices to assume that the coordinate spaces are Polish, or even just any measurable spaces, by the Ionescu Tulcea theorem (see Shirayev, 1984, p. 247).

from the set of all finite subsets of $\Theta_{\vec{t}}$ to the set of cylinders in

$$P\left[B_t^0 \times \left(\bigtimes_{n=1}^{\infty} \prod_{\ell=1}^{N+1} (\Theta_{\ell t}^n)\right)\right];$$

by weak convergence of measures, this map and its inverse are both continuous, and thus the map

$$f(\theta_{\vec{t}}): \bar{\Theta}_{\ell t} \rightarrow P\left[B_t^0 \times \left(\bigtimes_{n=1}^{\infty} \prod_{\ell=1}^{N+1} (\Theta_{\ell t}^n)\right)\right]$$

is a homeomorphism (see Brandenburger and Dekel, 1993, for detailed proof). However, this is not yet sufficient to ensure consistency of individual beliefs over beliefs of other players. For instance, it does not exclude the possibility that one of the players will act assuming the opponents are irrational. In equilibrium, this possibility is to be ruled out, which may be done in a number of ways. One way consists of defining a point of Ω_t — the universal belief space of γ_0 , — as the limit of beliefs held by coherent individuals (as done by Mertens and Zamir, 1985). Another way is by requiring other players' rationality to be common knowledge (as in Brandenburger and Dekel, 1993). Under this last assumption, attention has to be limited to a proper subset Ω_t of

$$B_t^0 \times \left(\bigtimes_{n=1}^{\infty} \prod_{\ell=1}^{N+1} (\Theta_{\ell t}^n)\right),$$

which satisfies both the coherence and common knowledge restrictions. This latter definition is easier and compatible with the general logic of our model; thus, we call

$$\Omega_t \subset B_t^0 \times \left(\bigtimes_{n=1}^{\infty} \prod_{\ell=1}^{N+1} (\Theta_{\ell t}^n)\right),$$

the *universal belief space* of the game γ_t . The Borel σ -algebra over that set shall be denoted by \mathfrak{B}_t . Showing that $\Omega_t \sim_{hmeo} P(\Omega_t) \equiv \Theta_t$, where Θ_t is the *universal types space* is also straightforward (Brandenburger and Dekel, 1993). Indeed, common knowledge implies that Θ_t is a set of all $\theta_{\vec{t}}$ s.t.

$$f_{\theta(\ell t)}\left[B_t^0 \times \left(\bigtimes_{n=1}^{\infty} \prod_{\ell=1}^{N+1} (\Theta_{\ell t}^n)\right)\right] = 1,$$

thus for every such $\theta_{\bar{t}}$, $f(\theta_{\bar{t}})$ is the set of elements of

$$P\left[B_t^0 \times \left(\bigtimes_{n=1}^{\infty} \prod_{\ell=1}^{N+1} (\Theta_{\ell t}^n)\right)\right],$$

for which the probability of event $B_t^0 \times (\Omega_t)^{N+1}$

equals one. But this last set is homeomorphic to Θ_t as the set of degenerate measures on a subspace of a metrizable space. This completes the construction of the universal belief and types spaces for the initial stage-game γ_0 .

Now we need to account for the dynamic aspect of the Ponzi game. This can be done if we notice that the universal belief space Ω_t contains all possible paths of the games γ_t , $t = 0, 1, 2, \dots$, where the observable part of a profile q_t "cuts off" those paths (subsets of Ω_t) that are known to not be played at each time period. These paths are exactly those which correspond to commonly known information that the firm did not defect at stage t . One may say that every move of the firm partitions the set of possible paths of the Ponzi game into two parts, corresponding to its cooperation or defection. This underlies the following construction of the conditional probabilities $\mu(H_{t+1}|G_t)$ of any possible event (set of paths) $H_{t+1} \subset \Omega_{t+1}$ with respect to any possible event $G_t \subset H_{t+1}$, which is known to contain (or not) the actual strategy q_{tN+1} played at t .

Every observed history observed at t forms a partition $\bar{n}_t = \{G_t, \neg G_t\}$ of Ω_t ; let the subset G_t of Ω_t correspond to the firm's cooperation at t . Then for any random variable μ_{t+1} on Ω_t for which a mathematical expectation is defined, the expected value of μ_{t+1} on G_t is also defined as

$$\int_{G(t)} \mu_{t+1}(\omega) \mu_t(d\omega), \quad \omega \in G_t.$$

This last expectation is countably additive, and thus is itself a measure, denoted by v_t and clearly, absolutely continuous with respect to μ_t : if $\mu_t(\omega) = 0$ then $v_t(\omega) = 0$. By the Radon–Nikodym theorem, we can write

$$\int_{G(t)} \mu_{t+1}(\omega) \mu_t(d\omega) = v_t(G_t) = \mu_t(H_{t+1} \cap G_t),$$

where $\mu_{t+1}(\cdot)$ is an (\bar{n}_t/\bar{o}_t) — measurable function. When μ_{t+1} is an indicator function of the set H_{t+1} , it is a *version of conditional probability*;³⁰

³⁰ This was also discussed by Mertens and Zamir, who have shown that consistency of players' beliefs (in the sense that for every subset A of Ω_t and every t , $\mu_t(A) = \int_{\omega \in \Omega} \mu_t(A|G_t) d\mu_t(G_t)$) is tantamount to saying they are Bayesians.

notably, its definition up to the set of measure 0 corresponds to the fact that it is approximately independent of the strategy of any single individual. The system $G_t \cap \mathfrak{d}_t$ is itself a σ -algebra over a smaller belief space $\Omega_{t+1} \subset \Omega_t$. Continuing iteratively, we obtain a decreasing sequence of belief subspaces $\Omega_0 \supset \Omega_1 \supset \Omega_2 \supset \dots$, all of which are compact Polish and serve as universal belief spaces for the corresponding stage games.

Building an extension for the continuation game to the next stage-game given the above sequence of universal belief spaces still remains to be done. This can be achieved if we restrict attention to those subsets of Ω_t that are not precluded by history $h_{\bar{t}}$, and introduce a Borel σ -algebra \mathfrak{d}_{t+1} on Ω_{t+1} . As shown above, in period $t+1$ each individual player F should select a (pure) strategy that maximizes his or her payoff at that stage;

$$e_{F,t+1} \in \operatorname{argmax}_{\theta_{-t}} \int_{\Omega_{t+1}} [e_{F,t+1}, q_{-F,t+1}, s | \theta, h_{\bar{t}}] \mu_{\bar{t}+1}(d\theta_{-F} s | h_{\bar{t}}), \quad (\text{A5})$$

where $\mu_{\bar{t}+1}$ and $h_{\bar{t}}$ are his or her beliefs at $t+1$ and information vector at t , and $q_{t+1} = (q_{F,t+1}, q_{N+1,t+1})$ is the profile to be played in the upcoming stage. A corresponding problem for the firm is written as stipulated in Section 3:

$$e_{N+1,t+1} \in \operatorname{argmax}_{\theta_{-t}} \int_{\Omega_{t+1}} [e_{N+1,t+1}, q_{-F,t+1}, s | \theta, h_{N+1,t}] \times \\ \times \mu_{N+1,t+1}(d\theta_{-F} s | \theta_{N+1}, h_{N+1,t}). \quad (\text{A6})$$

Note that our setup allows us to avoid conditioning this optimization problem on the expected strategies of every other player: all information is contained in the aggregates. Rules (5a) and (6a) determine those policies of player F that are not precluded by the profile q_t played at any point of time. A Cartesian product of policies compatible with the history of plays is essentially a (closed) subset of Ω_t , and it is made compatible by construction with the transition function $\rho_{s',s,t}(\cdot)$ for every t . By induction, a required sequence of types $\{\theta_t\}$, physical uncertainty $\{B_t\}$, and universal belief spaces Ω_t that are a compact and Polish space, endowed with σ -algebras $\{\mathfrak{d}_t\}$ is then obtained, and each member of this sequence indexed by t represents beliefs and physical uncertainty at that stage game. These sequences potentially extend to infinity, although in practice they are interrupted by the first defection of the firm. Q.E.D.

Proof of Proposition 5

Suppose that one (individual) player, observing $h_{\bar{t}}$ at t , excludes the set $\neg G_t$, and another (individual) player observes $h'_{\bar{t} \neq \bar{t}}$ and excludes

$\neg G'_t \neq \neg G_t$. At the next stage, systems $G_t \cap \partial_t$ and $G'_t \cap \partial_t$ will then be different, generating different belief spaces Ω_{t+1} and Ω'_{t+1} , and in particular, different physical uncertainty spaces B^1 . Taking $\cup^{N+1} B_{1F}$ as the basic derived space would not help, for then beliefs of player F are undefined over those points of that space which do not belong to the support of his or her type θ_{F1} — in other words, the types will fail to be *coherent*.

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